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AN EQUILIBRIUM FINITE ELEMENT METHOD FOR FOURTH ORDER ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT : An equilibrium finite element method for fourth order elliptic problems with variable coefficients on convex polygonal domains has been developed. In the particular case of bending problems of elastic anisotropic/orthotropic/isotropic plates with variable/constante thickness, this new equilibrium method allows a simultaneous approximation to the displacement and the bending and twisting moment tensor. Error estimates and results of numerical experimentes and results of numerical experiments are also given.

RESUME : Pour les problèmes elliptiques du quatrième ordre avec les coefficients variables sur un convexe polygonal, on a developpé une méthode équilibre des éléments finis. Dans le cas particulier des problèmes flexions des plaques élastiques anisotropiques/orthotropiques/isotropiques avec l'épaisseur variable/constant. Cette nouvelle méthode équilibre donne une approximation simultané de la deflection et du tenseur de moments. L'estimation d'erreur et les résultats des essayages numériques sont aussi données.

AN EQUILIBRIUM FINITE ELEMENT METHOD FOR FOURTH ORDER ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

P.K. Bhattacharyya*, S. Gopalsamy**

1. INTRODUCTION

In [5] - [8] four different mixed finite element methods have been developed for the Dirichlet problem of fourth order elliptic partial differential equations with variable/constant coefficients, the biharmonic problem and the bending problems of elastic -anisotropic/orthotropic/isotropic- plates with variable/constant thickness being particular cases of this problem, for which the mixed methods of [5] - [6] give a simultaneous approximation to displacement u, curvature tensor $(u_{,ij})$ and bending and twisting moment tensor (ψ_{ij}), and those of [7]-[8] allow a simultaneous approximation to displacement u and bending and twisting moment tensor (ψ_{ij}). In the mixed method of [8] , the continuity of the normal bending moment M_n at the interelement boundaries is required as in the mixed method of Hellan-Hermann-Johnson for the biharmonic problem [4],[10],[13],[17],[18],[20],[24],[25] . In [15] Fraeijs de Veubeke developed the equilibrium finite element method in which the continuity of both normal bending moment M_n and Kirchhoff transverse force K_n is required at the interelement boundaries, [16] being a subsequent paper based on the general procedure developed in [15] . Hence, the equilibrium finite element method may be regarded as a natural extension of the Hellan-Hermann-Johnson mixed method for the biharmonic problem [10],[11],[25].

The present paper is devoted to the development of such an equilibrium finite element method for the Dirichlet problem of fourth order partial differential equations with variable/constant coefficients. This paper contains new results in this direction. In fact, for bending problems of anisotropic/orthotropic/ isotropic plates with variable/constant thickness, this new equilibrium finite

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element method gives also a simultaneous approximation to displacement u and bending and twisting moment tensor (ψ_{ij}). Moreover, the equilibrium method [11],[25] for the biharmonic problem, which allows a simultaneous approximation to displacement u, change in curvature tensor $(u_{,ij})$ (but not the 'actual' bending and twisting moment tensor (ψ_{ij}) i.e. further computation will be necessary to find bending and twisting moments), can be retrieved as a particular case from the general scheme developed in this paper by means of a suitable choice of the coefficients of the equation. Under the assumption of necessary regularity of the solution of the Galerkin variational problem error estimates for the equilibrium finite element solution have been developed, following the general techniques of Brezzi-Raviart [10], Raviart [25], and Fortin [14].

2. NOTATIONS

Let Ω be a convex polygon with boundary Γ in $\mathbb{R}^2$ and $H^m(\Omega)$ be the Sobolev spaces [1],[23] of integral order $m \geq 0$ equipped with inner product $\langle .,. \rangle_{m,\Omega}$, norm $\|.\|_{m,\Omega}$ and seminorm $|.|_{m,\Omega}$ such that $H^o(\Omega) \equiv L^2(\Omega)$,

$$
\begin{aligned}
H_0^1(\Omega) &= \{v : v \in H^1(\Omega),\ \gamma_o v = v|_\Gamma = 0\}, \\
H_0^2(\Omega) &= \{v : v \in H^2(\Omega),\ \gamma_o v = v|_\Gamma = 0,\ \gamma_1 v = (\partial v/\partial n)|_\Gamma = 0\},
\end{aligned}
\tag{2.1}
$$

where $\partial v/\partial n$ is the derivative of v in the direction of the exterior normal to the boundary Γ;

$\gamma_k : H^m(\Omega) \to H^{m-k-1/2}(\Gamma)$ are trace operators [1],[23],

$m = 1,2$; $k = 0, m-1$; $H^{1/2}(\Gamma)$, $H^{3/2}(\Gamma)$ being the fractional order Sobolev spaces of functions on Γ ;

$H_0^m(\Omega) \equiv \overline{D(\Omega)}$ in the norm topology of $H^m(\Omega)$, $m \geq 0$,

$D(\Omega)$ being the space of test functions on Ω.

For $p > 2$, let $W_0^{1,p}(\Omega)$ be the Sobolev space [1],[23] defined by :

$$W_0^{1,p}(\Omega) = \{v : v \in W^{1,p}(\Omega),\ \gamma_o v = v|_\Gamma = 0\},$$

such that $\forall p > 2$, $H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ (2.2)

with dense, continuous injections, and $\forall p > 2$, $H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow C^o(\bar{\Omega})$, (2.3)

(2.2)-(2.3) being the consequences of the Sobolev's imbedding theorem [1],[12], [23] .

3. THE CONTINUOUS VARIATIONAL PROBLEM

To the Dirichlet problem (P) defined by : For given $f \in L^2(\Omega)$, find u such that

$$\Lambda u = f \quad \text{in} \quad \Omega \ , \ u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0, \tag{3.1}$$

where

$$(\Lambda u)(x) \equiv \frac{\partial^2}{\partial x_k \partial x_l}\left(a_{ijkl} \frac{\partial^2 u}{\partial x_i \partial x_j}\right)(x) \equiv (a_{ijkl}\, u_{,ij})_{,kl}(x) \ \text{in} \ \Omega \ , \tag{3.2}$$

(in (3.2) and also in the sequel, the Einstein's summation convention has been followed), we associate the following Galerkin variational problem (P_G) defined by :

$$\begin{aligned} &\text{Find } u \in H_0^2(\Omega) \quad \text{such that} \\ &a(u,v) = l(v) \quad \forall v \in H_0^2(\Omega) \ , \end{aligned} \tag{3.3}$$

where the continuous, symmetric, bilinear form a(.,.) and the continuous linear form l(.) are defined by : $\forall v, w \in H_0^2(\Omega)$

$$a(v,w) = \langle \Lambda v, w \rangle_{0,\Omega} = \int_\Omega a_{ijkl}\, v_{,ij}\, w_{,kl}\, d\Omega \ ; \tag{3.4}$$

$$l(v) = \langle f,v \rangle_{0,\Omega} = \int_\Omega fv \, d\Omega \qquad \forall v \in H_0^2(\Omega) \ ; \tag{3.5}$$

the coefficients a_{ijkl} satisfy the following conditions : $\forall i,j,k,l = 1,2,$

(A1) : $a_{ijkl} \in C^o(\bar{\Omega})$; $a_{ijkl}(x) \geq 0$; $a_{ijkl}(x) = a_{klij}(x) = a_{jikl}(x) =$

$a_{jilk}(x) \quad \forall x \in \bar{\Omega}$,

(A2) : $\forall \xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\exists$ a constant $\alpha_o > 0$ such that

$$\forall x \in \bar{\Omega}, \ a_{ijkl}(x)\, \xi_{ij}\xi_{kl} \geq \alpha_o \ \|\xi\|^2_{\mathbb{R}^4} \ ;$$

(A2') : $\forall v \in H_0^2(\Omega)$, $a(v,v) \geq \alpha \|v\|^2_{2,\Omega}$ for some $\alpha > 0$,

i.e. $a(.,.)$ defined by (3.4) is $H_0^2(\Omega)$-elliptic.

Remark (3.1) : In [9], sufficient conditions for (A2) to hold can be found along with interesting examples, and it has also been shown that (A2) => (A2').

Theorem (3.1) : Under the assumptions (A1)-(A2'), the problem (P_G) has a unique solution.

4. EQUILIBRIUM METHOD FORMULATION

For the construction of suitable bilinear forms of the equilibrium method to be formulated, we prepare the following results.
To every $\xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, we associate $\bar{\xi} = (\xi_{11},\xi_{22},\xi_{12}) \in \mathbb{R}^3$ such that $\forall x \in \bar{\Omega}$,

$$a_{ijkl}(x)\ \xi_{ij}\xi_{kl} = \bar{\xi}[A(x)]\ \bar{\xi}^T \geq \alpha_o\ \|\xi\|^2_{\mathbb{R}^4} \geq \alpha_o\ \|\bar{\xi}\|^2_{\mathbb{R}^3}\ ,$$

where a_{ijkl} satisfy (A1)-(A2) ; $\bar{\xi}^T$ is the transpose of $\bar{\xi}$; $[A(x)] \in \mathcal{L}(\mathbb{R}^3)$ is defined by : $\forall x \in \bar{\Omega}$,

$$[A(x)] = \begin{bmatrix} a_{1111}(x) & a_{1122}(x) & 2a_{1112}(x) \\ a_{2211}(x) & a_{2222}(x) & 2a_{2212}(x) \\ 2a_{1211}(x) & 2a_{1222}(x) & 4a_{1212}(x) \end{bmatrix} = [A(x)]^T . \tag{4.1}$$

Proposition (4.1) : $\forall x \in \bar{\Omega}, [A(x)] \in \mathcal{L}(\mathbb{R}^3)$, defined by (4.1) is symmetric, positive-definite, and its inverse $[A^{-1}(x)] \in \mathcal{L}(\mathbb{R}^3)$ defined by :
$\forall x \in \bar{\Omega}\ \ [A(x)]\,[A^{-1}(x)] = I$;

$$[A^{-1}(x)] = \begin{bmatrix} A_{1111}(x) & A_{1122}(x) & A_{1112}(x) \\ A_{2211}(x) & A_{2222}(x) & A_{2212}(x) \\ A_{1211}(x) & A_{1222}(x) & A_{1212}(x) \end{bmatrix} = [A^{-1}(x)]^T , \tag{4.2}$$

where $I \in \mathcal{L}(\mathbb{R}^3)$ is the identity matrix ; for $1 \leq i \leq j \leq 2$, $1 \leq k \leq l \leq 2$, $A_{ijkl} = A_{klij}$, is also symmetric, positive-definite.

Proposition (4.2) : $\forall x \in \bar{\Omega}$, the symmetric $[A^*(x)] \in \mathcal{L}(\mathbb{R}^3)$ defined by :

$$[A^*(x)] = \begin{bmatrix} A_{1111}(x) & A_{1122}(x) & 2A_{1112}(x) \\ A_{2211}(x) & A_{2222}(x) & 2A_{2212}(x) \\ 2A_{1211}(x) & 2A_{1222}(x) & 4A_{1212}(x) \end{bmatrix} = [A^*(x)]^T \tag{4.3}$$

is positive-definite, i.e. $\forall x \in \bar{\Omega}$, $\forall \bar{\xi} = (\xi_{11},\xi_{22},\xi_{12}) \in \mathbb{R}^3$, $\exists\, \alpha_1 > 0$ such that $\bar{\xi}\,[A^*(x)]\bar{\xi}^T \geq \alpha_1 \|\bar{\xi}\|^2_{\mathbb{R}^3}$. (4.4)

Proof : The symmetric $[A^*(x)]$ will be positive-definite, iff $\forall x \in \bar{\Omega}$,

$$A_{1111}(x) > 0, \quad \begin{vmatrix} A_{1111}(x) & A_{1122}(x) \\ A_{2211}(x) & A_{2222}(x) \end{vmatrix} > 0 \;;$$

$\det([A^*(x)]) > 0$, which follow from the Proposition (4.1), since $[A^{-1}(x)]$ is positive-definite, and $\det([A^*(x)]) = 4 \det([A^{-1}(x)]) > 0 \;\; \forall x \in \bar{\Omega}$.

Now, define new functions A_{2121}, A_{ij21}, A_{21ij} $(1 \leq i \leq j \leq 2)$ with the help of functions A_{ijkl} in (4.2) as follows : $A_{2121} = A_{1212}$, $A_{ij21} = A_{21ij} = A_{ij12}$, $1 \leq i \leq j \leq 2$, from which, together with (4.2), we have : $\forall x \in \bar{\Omega}$,

$$A_{ijkl}(x) = A_{klij}(x) = A_{lkij}(x) = A_{lkji}(x) \quad \forall i,j,k,l = 1,2. \tag{4.5}$$

Proposition (4.3) : $\forall x \in \bar{\Omega}$, $\forall \xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\forall \zeta = (\zeta_{11},\zeta_{22},\zeta_{12},\zeta_{21}) \in \mathbb{R}^4$ with $\zeta_{12} = \zeta_{21}$,

$$A_{ijkl}(x)\; a_{ijmn}(x)\; \xi_{mn}\, \zeta_{kl} = \xi_{ij}\, \zeta_{ij} \;; \tag{4.6}$$

$$A_{ijkl}(x)\; \xi_{ij}\, \xi_{kl} \geq \alpha_2 \|\xi\|^2_{\mathbb{R}^4} \quad \text{for some } \alpha_2 > 0 \tag{4.7}$$

Proof : From (4.1) and (4.2), $\forall x \in \bar{\Omega}$,

$[A^{-1}(x)][A(x)] = I \Rightarrow \forall x \in \bar{\Omega}$; $k,l = 1,2,$

$$A_{iikl}(x)\; a_{iikl}(x) = 1 \;; \tag{4.8}$$

$$2A_{12kl}(x)\; a_{12kl}(x) = 1 \;;$$

$$A_{ijkl}(x)\; a_{mnkl}(x) = 0 \text{ for } i \neq m \text{ or } j \neq n,$$

$$1 \leq i \leq j \leq 2,\; 1 \leq m \leq n \leq 2.$$

Then, from the property of symmetry of the coefficients a_{ijkl} and A_{ijkl} in (A1) and (4.5) respectively and from the relations (4.8), we have : $\forall x \in \bar{\Omega}$,

$$\begin{aligned} A_{ijkl}(x)\, a_{ijmn}(x)\, \xi_{mn}\, \zeta_{kl} = {} & (A_{ij11}(x)\, a_{ij11}(x)\, \xi_{11} + A_{ij11}(x)\, a_{ij22}(x)\, \xi_{22} \\ & + 2A_{ij11}(x)\, a_{ij12}(x)\, \xi_{12})\, \zeta_{11} + (A_{ij22}(x)\, a_{ij11}(x)\, \xi_{11} \\ & + A_{ij22}(x)\, a_{ij22}(x)\, \xi_{22} + 2A_{ij22}(x)\, a_{ij12}(x)\, \xi_{12})\, \zeta_{22} \\ & + 2(A_{ij12}(x)\, a_{ij11}(x)\, \xi_{11} + A_{ij12}(x)\, a_{ij22}(x)\, \xi_{22} \\ & + 2A_{ij12}(x)\, a_{ij12}(x)\, \xi_{12})\zeta_{12} = \xi_{11}\zeta_{11} + \xi_{22}\zeta_{22} + 2\xi_{12}\zeta_{12} \\ & = \xi_{ij}\ \zeta_{ij}\ . \end{aligned}$$

Now, for (4.7), we have : $\forall x \in \bar{\Omega}$,

$$A_{ijkl}(x)\ \xi_{ij}\ \xi_{kl} = \bar{\xi}\ [A^*(x)]\ \bar{\xi}^T \geq \alpha_1\ \|\bar{\xi}\|^2_{\mathbb{R}^3} \geqslant \alpha_2\ \|\xi\|^2_{\mathbb{R}^4}$$

with $\alpha_2 = \alpha_1/2$ (by virtue of (4.4),

$\forall \xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\forall \bar{\xi} = (\xi_{11},\xi_{22},\xi_{12}) \in \mathbb{R}^3$.

Remark (4.1) : $\forall i,j,k,l = 1,2$, the functions $A_{ijkl} \in C^o(\bar{\Omega})$, which follows from (A1) and the Proposition (4.1).

Remark (4.2) : From (A1), $a_{ijkl}(x) \geq 0$ $\forall x \in \bar{\Omega}$ $\forall i,j,k,l = 1,2$, but A_{ijkl} are not positive-valued functions in general, i.e. there may exist some $i,j,k,l = 1,2$ and some $x \in \bar{\Omega}$ such that $A_{ijkl}(x) < 0$.

Now, we define the following spaces :

$$\underline{H} = \{\Phi\ :\ \Phi = (\phi_{ij}),\ 1 \leq i,j \leq 2,\ \phi_{ij} \in L^2(\Omega),\ \phi_{12} = \phi_{21}\} \tag{4.9}$$

with $\|\Phi\|^2_{\underline{H}} = \|\Phi\|^2_{0,\Omega} = \sum_{i,j=1}^{2} \|\phi_{ij}\|^2_{0,\Omega}$ $\forall \Phi \in \underline{H}$;

$$W = W_o^{1,p}(\Omega),\ p > 2\ ;\ \|\chi\|_W = \|\chi\|_{1,p,\Omega} \quad \forall \chi \in W \tag{4.10}$$

Let T_h be an admissible triangulation [12] of $\bar{\Omega}$ into closed triangles T. Given a triangle $T \in T_h$ with vertices $\{a_{i,T}\}_{i=1}^{3}$ and boundary ∂T and a tensor-valued function $\Phi = (\phi_{ij})_{i,j=1,2}$ with $\phi_{ij} \in H^2(T)$ and $\phi_{ij} = \phi_{ji}$, $1 \leq i,j \leq 2$, we define the 'normal bending moment $M_n(\Phi)$', 'twisting moment $M_{nt}(\Phi)$', 'transverse shear force $Q_n(\Phi)$', 'Kirchhoff transverse force $K_n(\Phi)$' corresponding to the bending moment field Φ along ∂T as follows [25] :

$$
\begin{aligned}
&M_n(\Phi) = \phi_{ij}\, n_i n_j \;;\; M_{nt}(\Phi) = \phi_{ij}\, n_i t_j \;;\; Q_n(\Phi) = \phi_{ij,i}\, n_j \;; \\
&K_n(\Phi) = \frac{\partial M_{nt}(\Phi)}{\partial t} + Q_n(\Phi) ,
\end{aligned}
\tag{4.11}
$$

where $\vec{n} = (n_1, n_2)$ is the unit exterior normal and
$\vec{t} = (t_1, t_2) = (n_2, -n_1)$ is the unit tangent along ∂T.

Then, we have the following Green's formulae :

$\forall \phi_{ij} \in H^2(T)$, $1 \leq i,j \leq 2$, $\phi_{12} = \phi_{21}$, $\forall \chi \in H^2(T)$,

$$
\int_T \phi_{ij}\, \chi_{,ij}\, dT = -\int_T \phi_{ij,j}\, \chi_{,i}\, dT + \int_{\partial T} \left(M_n(\Phi)\frac{\partial \chi}{\partial n} + M_{nt}(\Phi)\frac{\partial \chi}{\partial t}\right) ds \;; \tag{4.12}
$$

$$
= \int_T \phi_{ij,ij}\, \chi\, dT + \int_{\partial T} \left(M_n(\Phi)\frac{\partial \chi}{\partial n} + M_{nt}(\Phi)\frac{\partial \chi}{\partial t} - Q_n(\Phi)\, \chi\right) ds \;; \tag{4.13}
$$

$$
\begin{aligned}
&= \int_T \phi_{ij,ij}\chi\, dT + \int_{\partial T} \left(M_n(\Phi)\frac{\partial \chi}{\partial n} - K_n(\Phi)\chi\right) ds \\
&+ \sum_{i=1}^{3} J_{i,T}(\Phi)\chi\,(a_{i,T}),
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\text{where } J_{i,T}(\Phi) &= M_{nt}(\Phi)(a^{+}_{i,T}) - M_{nt}(\Phi)(a^{-}_{i,T}) \\
&= \text{Jump of } M_{nt}(\Phi) \text{ at the vertex } a_{i,T},\ 1 \leq i \leq 3.
\end{aligned}
\tag{4.15}
$$

<u>Definition</u> : Let $\Phi \in \underline{H}$ with $\phi_{ij}|_T \in H^2(T)\ \forall T \in T_h$, $1 \leq i,j \leq 2$. Then, $M_n(\Phi)$ (resp. $K_n(\Phi)$) defined in (4.11) is said to be '<u>continuous at the interelement boundaries</u>' of the triangulation T_h, if and only if for any pair T_1, T_2 of adjacent triangles of T_h with a common side $T_1 \cap T_2$,

$$M_{n_1}(\Phi|_{T_1}) = M_{n_2}(\Phi|_{T_2}) \text{ on } T_1 \cap T_2 = \partial T_1 \cap \partial T_2 \tag{4.16}$$

$$(\text{resp. } K_{n_1}(\Phi|_{T_1}) = - K_{n_2}(\Phi|_{T_2}) \text{ on } T_1 \cap T_2 = \partial T_1 \cap \partial T_2) \tag{4.17}$$

where $\vec{n}_i = (n_1^i, n_2^i)$ is the unit exterior normal to the boundary ∂T_i of T_i, $i=1,2$.

Now, we can define the admissible space of moment fields of the equilibrium method formulation as follows :

$$\underline{V} = \{\Phi : \Phi \in \underline{H},\ \phi_{ij}|_T \in H^2(T)\ \forall T \in T_h,\ 1 \le i,j \le 2,\ M_n(\Phi)$$

$$\text{and } K_n(\Phi) \text{ are continuous at the interelement boundaries}\} \quad , \tag{4.18}$$

$$||\Phi||^2_{\underline{V}} = \sum_{i,j=1}^{2} \sum_{T \in T_h} ||\phi_{ij}||^2_{2,T} .$$

Obviously, we have $\underline{V} \hookrightarrow \underline{H}$, $W \hookrightarrow H_0^1(\Omega)$ with continuous imbeddings such that

$$\forall \Phi \in \underline{V}, \quad ||\Phi||_{\underline{H}} \le ||\Phi||_{\underline{V}} \ ; \tag{4.19}$$

$$\forall \chi \in W, \quad ||\chi||_{1,\Omega} \le \sigma_o \ ||\chi||_W \ \text{ for some } \sigma_o > 0 \ . \tag{4.20}$$

Proposition (4.4) : $\Phi = (\phi_{ij})_{1 \le i,j \le 2}$, $\phi_{12} = \phi_{21}$, $\phi_{ij} \in H^2(\Omega)$ $\forall i,j = 1,2 \Rightarrow \Phi \in \underline{V}$.

Proof : $\forall i,j = 1,2$, $\phi_{ij} \in H^2(\Omega) \Rightarrow \phi_{ij|T} \in H^2(T)$ $\forall T \in T_h$.

Again, $\forall i,j = 1,2$ $\phi_{ij} \in H^2(\Omega) \Rightarrow$ (i) $\phi_{ij} \in C^o(\bar{\Omega})$; (4.21)

(ii) $\dfrac{\partial \phi_{ij}}{\partial x_k}$, $\dfrac{\partial \phi_{ij}}{\partial t}$ are defined at the interelement boundaries of T_h ;

$$\text{(iii)} \ \frac{\partial}{\partial t} M_{n_1 t}(\Phi|_{T_1}) = - \frac{\partial}{\partial t} M_{n_2 t}(\Phi|_{T_2}),$$

$$Q_{n_1}(\Phi|_{T_1}) = - Q_{n_2}(\Phi|_{T_2}), \tag{4.22}$$

where T_1 and T_2 are any pair of adjacent triangles of T_h with the common side

$\partial T_1 \cap \partial T_2$, $\vec{n}_i$ being the unit exterior normal to the boundary ∂T_i of T_i $(i=1,2)$. Then, the continuity of $M_n(\Phi)$ and $K_n(\Phi)$ at the interelement boundaries of T_h follows from (4.21) and (4.22) respectively, and the result follows from the definition of $\underline{V}$.

Corollary (4.1) : (i) $\forall \Phi = (\phi_{ij}) \in \underline{V}$, $\forall \chi \in H^2(\Omega)$,

$$\int_\Omega \phi_{ij} \, \chi_{,ij} \, d\Omega = \sum_{T \in T_h} \int_T \phi_{ij,ij} \, \chi \, dT + \int_\Gamma (M_n(\Phi) \frac{\partial \chi}{\partial n} - K_n(\Phi)\chi) \, ds + \sum_{\substack{a_{i,T} \in N_h, \\ T \in T_h}} J_{i,T}(\Phi)\chi(a_{i,T}) ,$$

(ii) $\forall \Phi = (\phi_{ij}) \in \underline{V}$, $\forall \chi \in H_o^2(\Omega)$, (4.23)

$$\int_\Omega \phi_{ij} \, \chi_{,ij} \, d\Omega = \sum_{T \in T_h} \int_T \phi_{ij,ij} \, \chi \, dT + \sum_{\substack{a_{i,T} \in N_{oh}, \\ T \in T_h}} J_{i,T}(\Phi)\chi(a_{i,T}) ,$$

where N_h (resp. N_{oh}) denote the set of all vertices (resp. all interior vertices) of triangles of T_h in $\bar{\Omega}$ (resp. Ω).

Now we define the continuous bilinear forms :

$A(.,.) : \underline{H} \times \underline{H} \longrightarrow \mathbb{R}$; $b(.,.) : \underline{V} \times W \longrightarrow \mathbb{R}$ as follows :

$$\forall \Psi = (\psi_{ij}),\ \Phi = (\phi_{ij}) \in \underline{H}, \quad A(\Psi,\Phi) = \int_\Omega A_{ijkl} \, \psi_{ij} \phi_{kl} \, d\Omega = A(\Phi,\Psi), \tag{4.24}$$

where $A_{ijkl} = A_{ijkl}(x)$ are defined in (4.1)-(4.8) ;

$$b(\Phi,\chi) = - \sum_{T \in T_h} [\int_T \phi_{ij,ij} \, \chi \, dT + \sum_{i=1}^{3} J_{i,T}(\Phi) \, \chi(a_{i,T})] , \tag{4.25}$$

where $J_{i,T}(\Phi)$ are defined in (4.15).

Proposition (4.5) : $A(.,.)$ and $b(.,.)$ defined by (4.24) and (4.25) are continuous on $\underline{H} \times \underline{H}$ and $\underline{V} \times W$ respectively, i.e. $\exists$ constants $M > 0$, $m^* > 0$ such that

$$|A(\Psi,\Phi)| \leq M \|\Psi\|_{0,\Omega} \|\Phi\|_{0,\Omega} \quad \forall \Psi, \Phi \in \underline{H}, \tag{4.26}$$

$$|b(\Phi,\chi)| \leq m^* \|\Phi\|_{\underline{V}} \, \|\chi\|_W \quad \forall \Phi \in \underline{V}, \ \forall \chi \in W. \tag{4.27}$$

<u>Theorem (4.1)</u> : (i) $A(.,.)$ defined by (4.24) is $\underline{H}$-elliptic, i.e.
$\exists\ \alpha_2 > 0$ such that $\forall \Phi \in \underline{H}$, $A(\Phi,\Phi) \geq \alpha_2 \|\Phi\|^2_{0,\Omega}$; (4.28)

(ii) $\exists \beta > 0$ such that $\forall \chi \in W$,

$$\sup_{\Phi \in \underline{V}} \frac{b(\Phi,\chi)}{\|\Phi\|_{\underline{V}}} \geq \beta \|\chi\|_{0,\Omega} \quad . \tag{4.29}$$

<u>Proof</u> : (i) From (4.24) and (4.7), we have : $\forall \Phi = (\phi_{ij}) \in \underline{H}$, $A(\Phi,\Phi) =$

$$\int_\Omega A_{ijkl}(x) \quad \phi_{ij}(x)\ \phi_{kl}(x)$$

$$\geq \int_\Omega \alpha_2 \quad \phi_{ij}(x)\ \phi_{ij}(x)\ dx = \alpha_2 \|\Phi\|^2_{0,\Omega} \ .$$

(ii) The following proof is given in [25] .

Define $\Phi^* = (\mu\delta_{ij})$ with $\mu \in H^1_0(\Omega) \cap H^2(\Omega)$ [21] as the solution of the following Dirichlet problem on convex polygon Ω : For any $\chi \in W^{1,p}_0(\Omega)$, $p > 2$,

$$- \Delta\mu = \chi \text{ in } \Omega \quad , \mu|_\Gamma = 0,$$

and having the property : $\|\mu\|_{2,\Omega} \leq \tilde{c} \|\chi\|_{0,\Omega}$ for some $\tilde{c} > 0$ (4.30)

Then, from the Proposition (4.4), $\Phi^* \in \underline{V}$, and from (4.25), $\forall \chi \in W$,

$$\begin{aligned} b(\Phi^*,\chi) &= - \sum_{T \in T_h} \int_T (\mu\ \delta_{ij})_{,ij} \qquad dT = - \int_\Omega (\Delta\mu)\ \chi\ d\Omega \\ &= \|\chi\|^2_{0,\Omega}, \end{aligned} \tag{4.31}$$

since $M_{nt}(\Phi^*) = 0 \Rightarrow J_{i,T}(\Phi^*) = 0 \quad \forall a_{i,T} \in T,\ \forall T \in T_h$.

$$\text{But } \|\Phi^*\|^2_{\underline{V}} = \sum_{T \in T_h} \sum_{i,j=1,2} \|\mu\ \delta_{ij}\|^2_{2,T} = 2\ \|\mu\|^2_{2,\Omega} \Rightarrow \|\Phi^*\|_{\underline{V}} \tag{4.32}$$

$$\leq - \sqrt{2}\ \tilde{c}\ \|\chi\|_{0,\Omega} \quad \text{(from (4.30))}.$$

Then, from (4.31) and (4.32), we have

$$\sup_{\Phi \in \underline{V}} \frac{b(\Phi,\chi)}{\|\Phi\|_{\underline{V}}} \geq \frac{b(\Phi^*,\chi)}{\|\Phi^*\|_{\underline{V}}} \geq \beta \|\chi\|_{0,\Omega} \quad \text{with} \quad \beta = \frac{1}{\sqrt{2}\,\tilde{c}} > 0 .$$

Now, for the problem (P_G) defined in (3.3)-(3.5), we can construct the problem (Q) of the equilibrium method under consideration as follows :
Find $(\Psi,\lambda) \in \underline{V} \times W$ such that

$$A(\Psi,\Phi) + b(\Phi,\lambda) = 0 \quad \forall \Phi \in \underline{V} , \tag{4.33}$$

(Q)

$$b(\Psi,\chi) = - \langle f,\chi \rangle_{0,\Omega} \quad \forall \chi \in W, \tag{4.34}$$

where $A(.,.)$ and $b(.,.)$ are defined by (4.24) and (4.25) respectively.

<u>Theorem (4.2)</u> : The problem (Q) has at most one solution.

<u>Proof</u> : Let (Ψ_1,λ_1), $(\Psi_2,\lambda_2) \in \underline{V} \times W$ be any two solutions of (Q). Then, $(\Psi^*,\lambda^*) = (\Psi_1 - \Psi_2, \lambda_1 - \lambda_2) \in \underline{V} \times W$ satisfy the corresponding homogeneous system :
$A(\Psi^*,\Phi) + b(\Phi,\lambda^*) = 0 \quad \forall \Phi \in \underline{V}$,
$b(\Psi^*,\chi) = 0 \quad \forall \chi \in W$.

Then, $b(\Psi^*,\lambda) = 0 \Rightarrow A(\Psi^*,\Psi^*) = 0 \Leftrightarrow \Psi^* = 0$ in $\underline{H}$ by virtue of (4.28). Hence, $b(\Phi,\lambda^*) = 0 \quad \forall \Phi \in \underline{V} \Leftrightarrow \lambda^* = 0$ by (4.29).
Thus, $\Psi_1 = \Psi_2$, $\lambda_1 = \lambda_2$.
Since $A(.,.)$ is not a priori $\underline{V}$-elliptic, the problem (Q) is not well-posed in general, i.e. the existence of solution of (Q) cannot be proved in general.
But we have

<u>Theorem (4.3)</u> : If u be the solution of the problem (P_G) such that

$$u \in H_0^2(\Omega) \cap H^4(\Omega) \text{ and } a_{ijkl}\, u_{,kl} \in H^2(\Omega) \;\forall i,j = 1,2, \tag{4.35}$$

then $(\Psi,\lambda) = (\Psi,u) \in \underline{V} \times W$ with $\Psi = (\psi_{ij})_{1 \leq i,j \leq 2}$, $\psi_{ij} = a_{ijkl}\, \lambda_{,kl}$

is the solution of the problem (Q).

Conversely, if $(\Psi,\lambda) \in \underline{V} \times W$ be the solution of the problem (Q) with $\Psi = (\psi_{ij})$, $1 \le i,j \le 2$, then $\lambda = u \in H_0^2(\Omega)$ is the solution of the problem (P_G) and

$$\psi_{ij} = a_{ijkl}\, \lambda_{,kl} = a_{ijkl}\, u_{,kl} \quad \forall\, i,j = 1,2 \,. \tag{4.36}$$

<u>Proof</u> : Let $u \in H_0^2(\Omega) \cap H^4(\Omega)$ be the solution of (P_G) with $a_{ijkl}\, u_{,kl} \in H^2(\Omega)$ $\forall i,j = 1,2$, i.e.

$$\int_\Omega a_{ijkl}\, u_{,ij}\, v_{,kl}\, d\Omega = \langle f,v\rangle_{0,\Omega} \quad \forall v \in H_0^2(\Omega) \,. \tag{4.37}$$

Set $\lambda = u$; $\psi_{ij} = a_{ijkl}\, u_{,kl} = a_{ijkl}\, \lambda_{,kl} \in H^2(\Omega)$.

Then, from the Proposition (4.4), $\Psi \in \underline{V}$. Now, for $\lambda \in H_0^2(\Omega)$, $\forall\, \Phi \in \underline{V}$,

$$\begin{aligned} b(\Phi,\lambda) &= - \sum_{T \in T_h} \Big[\int_T \phi_{ij,ij}\, \lambda \; dT + \sum_{i=1}^{3} J_{i,T}(\Phi)\; \lambda(a_{i,T})\Big] \\ &= - \int_\Omega \phi_{ij}\; \lambda_{,ij}\; d\Omega \quad \text{(from (4.23))}. \end{aligned}$$

Hence, for $(\Psi,\lambda) \in \underline{V} \times W$ and $\forall \Phi \in \underline{V}$, we have

$$\begin{aligned} A(\Psi,\Phi) + b(\Phi,\lambda) &= \int_\Omega A_{ijkl}(x)\; \psi_{ij}(x)\; \phi_{kl}(x)\; d\Omega \\ &\quad - \int_\Omega \phi_{ij}(x)\; \lambda_{,ij}(x)\; d\Omega \\ &= \int_\Omega A_{ijkl}(x)\; a_{ijmn}(x)\; \lambda_{,mn}(x)\; \phi_{kl}(x)\; d\Omega \\ &\quad - \int_\Omega \phi_{ij}(x)\; \lambda_{,ij}(x)\, d\Omega \\ &= \int_\Omega \lambda_{,ij}(x)\; \phi_{ij}(x)\; d\Omega - \int_\Omega \lambda_{,ij}(x)\; \phi_{ij}(x)\; d\Omega \,, \\ &\equiv 0 \,. \qquad\qquad \text{(from (4.6))}. \end{aligned}$$

Thus, $(\Psi,\lambda) \in \underline{V} \times W$ satisfies (4.33).

Again, we have : $\forall \chi \in H_0^2(\Omega)$,

$$\begin{aligned} b(\Psi,\chi) &= - \int_\Omega \psi_{ij}\, \chi_{,ij}\; d\Omega = - \int_\Omega a_{ijkl}\, u_{,kl}\, \chi_{,ij}\; d\Omega \\ &= - \langle f,\chi\rangle_{0,\Omega} \qquad \text{(from (4.37))}. \end{aligned}$$

$\Rightarrow$ (4.34) holds for $\Psi \in \underline{V}$ and $\forall \chi \in H_0^2(\Omega)$.

Since $H_0^2(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, $p > 2$, (by virtue of (2.2)) and $b(\Psi,.)$ is continuous on W, (4.34) will also hold for all $\chi \in W_0^{1,p}(\Omega)$, $p > 2$, i.e.

$(\Psi,\lambda) \in \underline{V} \times W$ satisfies (4.34). Consequently, $(\Psi,\lambda) \in \underline{V} \times W$ is a solution of the problem (Q) and its uniqueness follows from the Theorem (4.2).

Conversely, let $(\Psi,\lambda) \in \underline{V} \times W$ be the solution of the problem (Q) with $\Psi = (\psi_{ij})$, $1 \le i,j \le 2$. Define $\Phi^* = (\phi\delta_{ij})_{1 \le i,j \le 2}$ with $\phi \in D(\Omega)$. Then, $\Phi^* \in \underline{V}$. Since $(\Psi,\lambda) \in \underline{V} \times W$ is the solution of (Q), we have from (4.33) : $\forall\phi \in D\ (\Omega)$,

$$A(\Psi,\Phi^*) + b(\Phi^*,\lambda) = 0$$

$$\Rightarrow \int_\Omega A_{ijkl} \ \psi_{ij}(\phi\delta_{kl})d\Omega - \int_\Omega (\phi\delta_{kl})_{,kl} \ \lambda \ d\Omega = 0 \quad \forall\phi \in D\ (\Omega)$$

$$\Rightarrow \int_\Omega (A_{ijkl} \ \psi_{ij} \ \delta_{kl})\phi \ d\Omega = \int_\Omega \lambda_{,i} \ \phi_{,i} \ d\Omega \quad \forall\phi \in H_0^1(\Omega)$$

since $D\ (\Omega)$ is dense in $H_0^1(\Omega)$.

Since $A_{ijkl}\ \psi_{ij}\ \delta_{kl} \in L^2(\Omega)$, Ω is a convex polygon, $\lambda \in H_0^1(\Omega) \cap H^2(\Omega)$ [21] with

$$\Delta\lambda = A_{ijkl} \ \psi_{ij} \ \delta_{kl} \ . \tag{4.38}$$

Again, choosing $\tilde{\Phi} = (\tilde{\phi}\ \delta_{ij})_{1 \le i,j \le 2}$ with $\tilde{\phi} \in D(\bar{\Omega})$, we have $\tilde{\Phi} \in \underline{V} \quad \forall\tilde{\phi} \in D(\bar{\Omega})$. Then, for $\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$, $\forall\tilde{\phi} \in D(\bar{\Omega})$, we have

$$b(\tilde{\Phi},\lambda) = -\sum_{T \in T_h} \int_T (\tilde{\phi}\delta_{ij})_{,ij} \ \lambda \ dT = -\int_\Omega (\Delta\tilde{\phi})\lambda \ d\Omega$$

$$= \int_\Omega \tilde{\phi}_{,i} \ \lambda_{,i} \ d\Omega = - A(\Psi,\tilde{\Phi})$$

$$= -\int_\Omega (A_{ijkl} \ \psi_{ij} \ \delta_{kl})\tilde{\phi} \ d\Omega$$

$$\Rightarrow \forall\tilde{\phi} \in H^1(\Omega), \ \int_\Omega \lambda_{,i} \ \tilde{\phi}_{,i} \ d\Omega = -\int_\Omega (A_{ijkl} \ \psi_{ij} \ \delta_{kl})\tilde{\phi} \ d\Omega$$

$$\Rightarrow \forall\tilde{\phi} \in H^1(\Omega), \ -\int_\Omega (\Delta\lambda)\tilde{\phi} \ d\Omega + \int_\Gamma \frac{\partial\lambda}{\partial n}\tilde{\phi} \ d\Gamma = -\int_\Omega (\Delta\lambda)\tilde{\phi} \ d\Omega \ \text{(from (4.38))}$$

$$\Rightarrow \langle\gamma_1 \lambda, \gamma_o\tilde{\phi}\rangle_{0,\Gamma} = 0 \quad \forall\tilde{\phi} \in H^1(\Omega)$$

$$\Rightarrow \gamma_1 \lambda = (\partial\lambda/\partial n)|_\Gamma = 0, \ \text{since } \gamma_1\lambda, \ \gamma_o\lambda \in H^{1/2}(\Gamma), \ \text{and } H^{1/2}(\Gamma) \equiv \gamma_o(H^1(\Omega)).$$

Then, $\lambda \in H_0^1(\Omega) \cap H^2(\Omega)$ with $\gamma_1\lambda = 0 \Rightarrow \lambda \in H_0^2(\Omega)$. Now, for $(\Psi,\lambda) \in \underline{V} \times W$ with $\lambda \in H_0^2(\Omega)$, we are to prove (4.36), i.e. $\psi_{ij} = a_{ijkl}\ \lambda_{,kl}$ $\forall i,j = 1,2$. For this, we have from (4.33) : For $(\Psi,\lambda) \in \underline{V} \times W$ with $\lambda \in H_0^2(\Omega)$, $\forall\Phi = (\phi_{ij}) \in \underline{V}$, $\int_\Omega A_{ijkl}\ \psi_{ij}\ \phi_{kl}\, d\Omega = \int_\Omega \phi_{ij}\ \lambda_{,ij}\, d\Omega$ (by (4.23))

$$\Rightarrow \int_\Omega (A_{ijkl}\ \psi_{ij} - \lambda_{,kl})\ \phi_{kl}\ d\Omega = 0 \quad \forall\Phi \in \underline{V}$$

Choose $\underset{\sim}{\Phi} = (\phi,0,0,0)$, $\tilde{\Phi} = (0,\phi,\phi,0)$, and $\underset{\approx}{\Phi} = (0,0,0,\phi)$ with $\phi \in \mathcal{D}(\Omega)$. Then $\underset{\sim}{\Phi}, \tilde{\Phi}, \underset{\approx}{\Phi} \in \underline{V}$ and we get : $\forall\phi \in \mathcal{D}(\Omega)$.

$$\int_\Omega (A_{ij11}\ \psi_{ij} - \lambda_{,11})\phi\ d\Omega = 0, \quad \int_\Omega (A_{ij22}\ \psi_{ij} - \lambda_{,22})\phi\ d\Omega = 0,$$

$$\int_\Omega [(A_{ij12}\ \psi_{ij} - \lambda_{,12}) + (A_{ij21}\ \psi_{ij} - \lambda_{,21})]\phi\ d\Omega = 0$$

$\Leftrightarrow$ $\forall k,l = 1,2$, $A_{ijkl}\ \psi_{ij} - \lambda_{,kl} = 0$, since $\forall k,l = 1,2$, $A_{ijkl}\ \psi_{ij} - \lambda_{,kl} \in L^2(\Omega)$

and $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$.

$$\Rightarrow \lambda_{,kl} = A_{ijkl}\ \psi_{ij} \quad \forall k,l = 1,2 \tag{4.39}$$

$\Rightarrow$ $\forall m,n = 1,2$, $a_{mnkl}\ \lambda_{,kl} = a_{mnkl}\ A_{ijkl}\ \psi_{ij} = \psi_{mn}$ by virtue of (4.8), and (4.36) is thus established.

It remains to show that $\lambda \in H_0^2(\Omega)$ satisfying (4.36) is the solution of the problem (P_G). For this, we have from (4.34), (4.36) and (4.23) : $\forall\chi \in H_0^2(\Omega)$,

$$b(\Psi,\chi) = -\int_\Omega \psi_{ij}\ \chi_{,ij} d\Omega = -<f,\chi>_{0,\Omega}$$

$$\Rightarrow \int_\Omega a_{ijkl}\ \lambda_{,kl}\ \chi_{,ij}\ d\Omega = <f,\chi>_{0,\Omega} \quad \forall\chi \in H_0^2(\Omega)$$

with $\lambda \in H_0^2(\Omega)$

$\Rightarrow$ $\lambda = u \in H_0^2(\Omega)$ is the solution of the problem (P_G) by the Theorem (3.1).

Remark (4.3) : If the 'change in curvature' tensor $(u_{,ij})$ is to be determined, then it can be easily done by using the formula (4.39), i.e. $\forall i,j = 1,2$, $u_{,ij} =$
$\lambda_{,ij} = A_{ijkl} \psi_{kl}$, (4.40)
where the components of the tensor $\Psi = (\psi_{ij})$ $1 \le i,j \le 2$ are 'bending and twisting moments' ; A_{ijkl} are defined by (4.1)-(4.8).

Example (4.1) : Choosing a_{ijkl} as follows : $a_{1111} = a_{2222} = 1$; $a_{1212} = a_{2121} = a_{2112} = a_{1221} = 1/2$, $a_{ijkl} = 0$ otherwise, which satisfy the assumptions (A1)-(A2) [9] , we get the biharmonic operator $\Lambda = \Delta\Delta$. From (4.1) and (4.2), the corresponding matrices $[A(x)]$ and $[A^{-1}(x)]$ are given by $\forall x \in \bar{\Omega}$,

$$[A(x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad ; \quad [A^{-1}(x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$ such

that $A_{ijkl} = a_{ijkl}$ $\forall i,j,k,l = 1,2$, and the bilinear form $A(.,.)$ in (Q) is given by :

$$A(\Psi,\Phi) = \int_\Omega \psi_{ij}\, \phi_{ij}\, d\Omega \quad \forall \Psi = (\psi_{ij}),\ \Phi = (\phi_{ij}) \in \underline{H} ,$$

$b(.,.)$ is defined by (4.25), since it is independent of the choice of a_{ijkl} (resp. A_{ijkl}). Then, the solution $(\Psi,\lambda) \in \underline{V} \times W$ of the problem (Q) in this particular case :

$$\int_\Omega \psi_{ij}\, \phi_{ij}\, d\Omega - \sum_{T\in T_h} \Big[\int_T \phi_{ij,ij}\, \lambda\, dT + \sum_{i=1}^{3} J_{i,T}(\Phi)\lambda(a_{i,T}) \Big] = 0 \qquad \forall \Phi \in \underline{V} ,$$

$$- \sum_{T\in T_h} \Big[\int_T \psi_{ij,ij}\, \chi\, dT + \sum_{i=1}^{3} J_{i,T}(\Psi)\chi(a_{i,T}) \Big] = - \langle f,\chi\rangle_{0,\Omega} \quad \forall \chi \in W,$$

which is the equilibrium formulation for the biharmonic equation [11], [25] is given by : $\lambda = u$, $\Psi = (\psi_{ij})$ with $\psi_{ij} = a_{ijkl}\, u_{,kl} = u_{,ij}$ $\forall i,j = 1,2$, where $u \in H_0^2(\Omega) \cap H^4(\Omega)$ is the solution of the problem (P_G) corresponding to the biharmonic equation.

If u denotes the deflection of the plate, then $\psi_{ij} = u_{,ij}$ are components of the change in curvature tensor, but not the 'actual' bending and twisting moments in the plate (see Example (4.3) and Remark (4.4)).

Example (4.2) : For the bending problems of clamped orthotropic plate with variable (or constant) thickness [9] ,[19],[22] we have

$$a_{iiii} = D_i \ ; \ a_{1122} = a_{2211} = \nu_1 D_2 = \nu_2 D_1 \ ; \ a_{1212} = a_{2121} = a_{2112} = a_{1221} = D_t \ ;$$

$$a_{ijkl} = 0 \text{ otherwise,} \tag{4.41}$$

and consequently,

$$\Lambda u \equiv (D_1 u_{,11} + \nu_1 D_2 u_{,22})_{,11} + 4(D_t u_{,12})_{,12} + (\nu_2 D_1 u_{,11} + D_2 u_{,22})_{,22} \ ,$$

where $D_i = E_i h^3/(12(1-\nu_1\nu_2)) > 0, \quad (i = 1,2)$;

$$D_t = G h^3/12 > 0, \ H = D_1 \nu_2 + 2 D_t \ ,$$

$$G = E_1 E_2/(E_1 + (1+ 2\nu_1)E_2) > 0,$$

$E_1 \nu_2 = E_2 \nu_1$, E_i and ν_i being Young's moduli and Poisson's coefficients respectively,

thickness function $h \in C^0(\bar{\Omega})$ such that

$0 < h_o \le h(x_1,x_2) \le h_1 \quad \forall (x_1,x_2) \in \bar{\Omega}$.

Then, from (4.1), the matrix $[A(x)]$ is as follows : $\forall x \in \bar{\Omega}$,

$$[A(x)] = \begin{bmatrix} D_1(x) & \nu_2 D_1(x) & 0 \\ \nu_1 D_2(x) & D_2(x) & 0 \\ 0 & 0 & 4D_t(x) \end{bmatrix} = [A(x)]^T \ .$$

From (4.41) ,it is obvious that a_{ijkl} satisfy the assumption (A1). For (A2), we have $\forall \xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\exists \ \alpha_o > 0$ such that $\forall x \in \bar{\Omega}$, $\forall \bar{\xi} = (\xi_{11},\xi_{22},\xi_{12}) \in \mathbb{R}^3$, Remark (4.3) : If the 'change in curvature' tensor $(u_{,ij})$ is to be determined, then it can be easily done by using the formula (4.39), i.e. $\forall i,j = 1,2$, $u_{,ij} = \lambda_{,ij} = A_{ijkl}\ \psi_{kl}$, (4.40)

where the components of the tensor $\Psi = (\psi_{ij})$ $1 \le i,j \le 2$ are 'bending and twisting moments' ; A_{ijkl} are defined by (4.1)-(4.8).

Example (4.1) : Choosing a_{ijkl} as follows : $a_{1111} = a_{2222} = 1$; $a_{1212} = a_{2121} = a_{2112} = a_{1221} = 1/2$, $a_{ijkl} = 0$ otherwise, which satisfy the assumptions (A1)-(A2) [9] , we get the biharmonic operator $\Lambda = \Delta\Delta$. From (4.1) and (4.2), the corresponding matrices $[A(x)]$ and $[A^{-1}(x)]$ are given by $\forall x \in \bar{\Omega}$,

$$[A(x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad ; \quad [A^{-1}(x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \quad \text{such}$$

that $A_{ijkl} = a_{ijkl}$ $\forall i,j,k,l = 1,2$, and the bilinear form $A(.,.)$ in (Q) is given by :

$$A(\Psi,\Phi) = \int_\Omega \psi_{ij}\ \phi_{ij}\ d\Omega \quad \forall\Psi = (\psi_{ij}),\ \Phi = (\phi_{ij}) \in \underline{H} ,$$

$b(.,.)$ is defined by (4.25), since it is independent of the choice of a_{ijkl} (resp. A_{ijkl}). Then, the solution $(\Psi,\lambda) \in \underline{V} \times W$ of the problem (Q) in this particular case :

$$\int_\Omega \psi_{ij}\ \phi_{ij}\ d\Omega - \sum_{T \in T_h} [\int_T \phi_{ij,ij}\ \lambda\ dT + \sum_{i=1}^{3} J_{i,T}(\Phi)\lambda(a_{i,T})] = 0 \quad \forall\Phi \in \underline{V} ,$$

$$- \sum_{T \in T_h} [\int_T \psi_{ij,ij}\ \chi\ dT + \sum_{i=1}^{3} J_{i,T}(\Psi)\chi(a_{i,T})] = - \langle f,\chi \rangle_{0,\Omega} \quad \forall\chi \in W,$$

which is the equilibrium formulation for the biharmonic equation [11], [25] is given by : $\lambda = u$, $\Psi = (\psi_{ij})$ with $\psi_{ij} = a_{ijkl}\ u_{,kl} = u_{,ij}$ $\forall i,j = 1,2$, where $u \in H_0^2(\Omega) \cap H^4(\Omega)$ is the solution of the problem (P_G) corresponding to the biharmonic equation.

If u denotes the deflection of the plate, then $\psi_{ij} = u_{,ij}$ are components of the change in curvature tensor, but not the 'actual' bending and twisting moments in the plate (see Example (4.3) and Remark (4.4)).

Example (4.2) : For the bending problems of clamped orthotropic plate with variable (or constant) thickness [9] ,[19],[22] we have.

$a_{iiii} = D_i$; $a_{1122} = a_{2211} = \nu_1 D_2 = \nu_2 D_1$; $a_{1212} = a_{2121} = a_{2112} = a_{1221} = D_t$;

$a_{ijkl} = 0$ otherwise,

and consequently, (4.41)

$$\Lambda u \equiv (D_1 u_{,11} + \nu_1 D_2 u_{,22})_{,11} + 4(D_t u_{,12})_{,12} + (\nu_2 D_1 u_{,11} + D_2 u_{,22})_{,22} ,$$

where $D_i = E_i h^3/(12(1-\nu_1\nu_2)) > 0$, $(i = 1,2)$;

$D_t = G h^3/12 > 0$, $H = D_1 \nu_2 + 2 D_t$,

$G = E_1 E_2/(E_1 + (1+ 2\nu_1)E_2) > 0$,

$E_1 \nu_2 = E_2 \nu_1$, E_i and ν_i being Young's moduli and Poisson's coefficients respectively,

thickness function $h \in C^0(\bar\Omega)$ such that

$0 < h_o \le h(x_1,x_2) \le h_1 \quad \forall (x_1,x_2) \in \bar\Omega$.

Then, from (4.1), the matrix $[A(x)]$ is as follows : $\forall x \in \bar\Omega$,

$$[A(x)] = \begin{bmatrix} D_1(x) & \nu_2 D_1(x) & 0 \\ \nu_1 D_2(x) & D_2(x) & 0 \\ 0 & 0 & 4D_t(x) \end{bmatrix} = [A(x)]^T .$$

From (4.41) it is obvious that a_{ijkl} satisfy the assumption (A1). For (A2), we have $\forall \xi = (\xi_{11},\xi_{22},\xi_{12},\xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\exists\ \alpha_o > 0$ such that $\forall x \in \bar\Omega$, $\forall \bar\xi = (\xi_{11},\xi_{22},\xi_{12}) \in \mathbb{R}^3$,

$$a_{ijkl}(x)\ \xi_{ij}\ \xi_{kl} = \bar{\xi}\ [A(x)]\ \bar{\xi}^T = D_1(x)\ \xi_{11}^2 + D_2(x)\ \xi_{22}^2 + 4\ D_t(x)\ \xi_{12}^2$$

$$+ 2\nu_1\ D_2(x)\ \xi_{11}\ \xi_{22}$$

$$\geq \alpha_o \|\xi\|^2_{\mathbb{R}^4} \geq \alpha_o\ \|\xi\|^2_{\mathbb{R}^3}\ ,$$

where $\alpha_o = \min\ \{\inf\limits_{x\in\bar{\Omega}} (D_1 - \nu_2 D_1)(x)\ ;\ \inf\limits_{x\in\bar{\Omega}} (D_2 - \nu_1\ D_2)(x)\ ;$

$\inf\limits_{x\in\bar{\Omega}}\ 2\ D_t(x)\} > 0$ [9] .

Thus, $\forall x \in \bar{\Omega}$, $[A(x)]$ is positive definite and its inverse $[A^{-1}(x)]$ is given by :

$$[\ A^{-1}(x)] = \frac{1}{|A(x)|} \begin{bmatrix} (4\ D_2 D_t)(x) & (-4\nu_2\ D_1 D_t)(x) & 0 \\ (-\ 4\nu_2\ D_1 D_t)(x) & (4\ D_1 D_t)(x) & 0 \\ 0 & 0 & D_1 D_2 (1-\nu_1\nu_2)(x) \end{bmatrix},$$

where $|A(x)| = \det[A(x)] = (4\ D_t D_1 D_2 (1-\nu_1\nu_2))(x)\ \forall x \in \bar{\Omega}$ such that from (4.2) and (4.5), we have : $A_{iiii} = 4\ D_j D_t / |A(.)|$ $(i \neq j)$, $A_{1122} = A_{2211} = -\ 4\nu_2 D_1 D_t / |A(.)|$ (see Remark (4.2)), $A_{1212} = A_{2112} = A_{1221} = D_1 D_2 (1-\nu_1\nu_2)/|A(.)|$, $A_{ijkl} = 0$ otherwise.

Consequently, $A_{ijkl} \in C^o(\bar{\Omega})$ $\forall i,j,k,l = 1,2$, and satisfy (4.6)-(4.8). Thus, for the orthotropic case, the problem (Q) is given by :

$$\int_\Omega \frac{1}{|A(x)|}\ [4\ D_2 D_t \psi_{11}\phi_{11} - 4\nu_2 D_1 D_t (\psi_{11}\phi_{22} + \psi_{22}\phi_{11})$$

$$+ 4\ D_1 D_t \psi_{22}\phi_{22} + 4\ D_1 D_2 (1-\nu_1\nu_2)\psi_{12}\phi_{12}]\ \ d\Omega$$

$$- \sum_{T\ \in T_h} [\int_T \phi_{ij,ij}\ \lambda\ \ dT + \sum_{i=1}^{3} J_{i,T}(\Phi)\ \lambda(a_{i,T})] = 0 \quad \forall\Phi \in \underline{V}\ ,$$

$$- \sum_{T\ \in T_h} [\int_T \psi_{ij,ij}\ \chi\ \ dT + \sum_{i=1}^{3} J_{i,T}(\Psi)\ \chi(a_{i,T})] = - \int_\Omega f\chi\ \ d\Omega\ \forall\chi \in W,$$

the solution $(\Psi,\lambda) \in \underline{V} \times W$ of which is characterized by : $\lambda =$ u is the deflection of the bent plate, $\Psi = (\psi_{ij})$ with $\psi_{ij} = a_{ijkl}\ u_{,kl}$ $(i,j = 1,2)$, ψ_{ij}'s being the 'actual' bending and twisting moments in the plate, i.e. $\psi_{11} = D_1(u_{,11} + \nu_2 u_{,22})$, $\psi_{22} = D_2(\nu_1 u_{,11} + u_{,22})$ are the bending moments in x_1- and x_2- directions, the twisting moment being $\psi_{12} = \psi_{21} = 2 D_t u_{,12}$. Then, the change in curvature tensor $(u_{,ij})$ is determined by (4.40) i.e. $u_{,11} = A_{11kl} \psi_{kl}$; $u_{,22} = A_{22kl} \psi_{kl}$, $u_{,12} = u_{,21} = A_{12kl} \psi_{kl}$. (4.42)

Example (4.3) : By putting $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$ and consequently, $D_1 = D_2 = D$ in all formulae for the orthotropic case in the Example (4.2), the corresponding formulae for the isotropic case are obtained, $\lambda = u$ being the deflection of the bent plate, $\psi_{11} = D(u_{,11} + \nu u_{,22})$, $\psi_{22} = D(\nu u_{,11} + u_{,22})$, $\psi_{12} = \psi_{21} = D(1-\nu)u_{,12}$ being bending moments in x_1- and x_2- directions and the twisting moment respectively. Then, the change in curvature tensor $(u_{,ij})$ is given by (4.42).

Remark (4.4) : In the equilibrium method of [11] , [25], for the isotropic case, the deflection $\lambda = u$ and the components $(u_{,ij})$ of the change in curvature tensor are obtained. To determine the 'actual' bending and twisting moments $(\psi_{ij})_{1 \le i,j \le 2}$, further computation is necessary using the formulae given in the Example (4.3).

Remark (4.5) : The anisotropic case [9],[22] can be dealt with exactly in the same way as for the orthotropic case in the Example (4.2).

5. FINITE ELEMENT APPROXIMATION :

Let $\{\lambda_i\}$ be the barycentric coordinates with respect to the vertices $\{a_{i,T}\}$ of a triangle $T \in \mathcal{T}_h$, ∂T being the boundary of T, $\mathcal{T}_h$ being the admissible triangulation introduced earlier. Let $P_k(T)$ be the linear space of restrictions to T of all polynomials in x_1 and x_2 and of degree $\le k$. Then, for $k \ge 2$ define the set of all "bubble functions" of $P_{k+1}(T)$, i.e.

$$B_{k+1}(T) = \{q : q \in P_{k+1}(T),\ q|_{\partial T} = 0\} = \{q : q = \lambda_1\lambda_2\lambda_3\, p,\ p \in P_{k-2}(T)\} \qquad (5.1)$$

is the set of all bubble functions of $P_{k+1}(T)$.

Remark (5.1) : If we would have let $k = 0,1$ in the definition of $B_{k+1}(T)$, then $B_{k+1}(T) = \emptyset$ for $k = 0,1$.

Proposition (5.1) : For $k \geq 2$, $B_{k+1}(T)$ defined by (5.1) is a linear space with dim $B_{k+1}(T)$ = dim $P_{k-2}(T)$.

For $k \geq 1$ we define the space of polynomials $P^*_{k+1} = P_1(T) + B_{k+1}(T)$ (5.2)

such that for $k = 1$, $P^*_2(T) = P_1(T)$, (5.3)

for $k = 2$, $P^*_3(T) = P_1(T) + \{\lambda_1\lambda_2\lambda_3\alpha\}$ (5.4)

with $\alpha \in \mathbb{R}$.

Now, we introduce the following finite dimensional spaces : For $k \geq 1$,

$$\underline{V}_h = \{\Phi_h : \Phi_h \in \underline{V},\ \Phi_h = (\phi_{hij})_{1 \leq i,j \leq 2},\ \phi_{hij}|_T \in P_k(T)\ \forall T \in T_h\} \tag{5.5}$$

$$W_h = \{\chi_h : \chi_h \in C^0(\bar{\Omega}),\ {}_h|_T \in P^*_{k+1}(T)\quad \forall T \in T_h,\ \chi_h|_\Gamma = 0\} \tag{5.6}$$

such that

$$\underline{V}_h \subset \underline{V} \subset \underline{H}\ , \quad W_h \subset W \subset H^1_0(\Omega). \tag{5.7}$$

Proposition (5.2) : (i) For $k = 1$,

dim W_h = total number of interior vertices in Ω

(ii) For $k \geq 2$,

dim W_h = total number of interior vertices in Ω +

(dim P_{k-2})$\times$ Number of triangles in $\bar{\Omega}$.

Now, we can construct the equilibrium finite element problem (Q_h) corresponding to the continuous problem (Q) as follows : Find $(\Psi_h,\lambda_h) \in \underline{V}_h \times W_h$ such that

$$(Q_h) \quad \begin{aligned} & A(\Psi_h,\Phi_h) + b(\Phi_h,\lambda_h) = 0 \qquad \forall \Phi_h \in \underline{V}_h, & (5.8)\\ & b(\Psi_h,\chi_h) = - \langle f,\chi_h\rangle_{0,\Omega} \qquad \forall \chi_h \in W_h, & (5.9)\end{aligned}$$

where $A(.,.)$ and $b(.,.)$ are defined by (4.24) and (4.25) respectively.

We make the following important assumption (external ellipticity condition or Babuska-Brezzi condition [13]) :

$$(A3) : \exists\ \beta_1 > 0 \text{ such that } \sup_{\Phi_h \in \underline{V}_h} \frac{|b(\Phi_h,\chi_h)|}{\|\Phi_h\|_{\underline{V}}} \geq \beta_1 \|\chi_h\|_W \qquad \forall \chi_h \in W_h$$

β_1 being independent of h.

Remark (5.2) : It will be shown later that for a specific choice of degrees of freedom for tensor-valued functions in $\underline{V}_h$ (see Proposition (5.3)), the assumption (A3) holds (see Proposition (5.4)).

Theorem (5.1) : Under the assumption (A3), the equilibrium finite element problem (Q_h) has a unique solution.

Proof : Since (Q_h) is defined on finite dimensional spaces $\underline{V}_h$, W_h, it is sufficient to prove that if $(\psi_h,\lambda_h) \in \underline{V}_h \times W_h$ be a solution of the corresponding homogeneous system :

$$A(\Psi_h,\Phi_h) + b(\Phi_h,\lambda_h) = 0 \quad \forall\ \Phi_h \in \underline{V}_h \tag{5.10}$$

$$b(\Psi_h,\chi_h) = 0 \quad \forall \chi_h \in W_h, \tag{5.11}$$

then $\Psi_h = 0$, $\lambda_h = 0$.

In fact, from (5.11), $b(\Psi_h,\lambda_h) = 0 \Rightarrow A(\Psi_h,\Psi_h) = 0 \Leftrightarrow \Psi_h = 0$ (by virtue of (4.28)).

Then, from (5.10), $b(\Phi_h,\lambda_h) = 0\ \forall\ \Phi_h \in \underline{V}_h \Leftrightarrow \lambda_h = 0$, which follows from (A3).

Since the existence and uniqueness of the solution of the problem (Q_h) have been established under the assumption (A3), now we shall show that (A3) holds, if we introduce the degrees of freedom for functions in $\underline{V}_h$ in a specific manner.

Let S_h be the set of sides in the triangulation T_h such that $T_i^* \in S_h$ $(1 \le i \le 3)$ are the three sides of a triangle $T \in T_h$.
$\forall T \in T_h$ define the following linear space $\underline{V}_T$ as follows

$$\underline{V}_T = \{\Phi_h : \Phi_h = (\phi_{hij})_{1 \le i,j \le 2},\ \phi_{h12} = \phi_{h21};\ \phi_{hij} \in P_k(T)\ \forall i,j = 1,2\} \tag{5.12}$$

where $P_k(T)$ is the linear space of restrictions to $T \in T_h$ of all polynomials

of degree $\leq k$ in variables x_1 and x_2, $k \geq 1$.
Then, $\dim \underline{V}_T = 3(k+2)(k+1)/2$.
Consider the set Σ_T of linearly independent linear functionals on $\underline{V}_T$ defined by : For $\Phi_h = (\phi_{hij}) \in \underline{V}_T$

$$\Sigma_T = \{ \int_{T_i^*} M_n(\Phi_h) q \, ds, \; q \in P(T_i^*), \; 1 \leq i \leq 3 \; ; \int_{T_i} K_n(\Phi_h) q \, ds, \; q \in P_{k-1}(T_i^*), \; 1 \leq i \leq 3, \int_T \phi_{hij} \, b_{,ij} \, p \, dT, \; b \in B_{k+1}(T), \; p \in P_1(T)\} \, , \tag{5.14}$$

$$\text{Card}(\Sigma_T) = 3(k+1) + 3k + 3k(k-1)/2 = 3(k+2)(k+1)/2 = \dim \underline{V}_T \tag{5.15}$$

<u>Lemma</u> (5.1) : For $1 \leq k \leq 3$, Σ_T defined by (5.14) is $\underline{V}_T$-unisolvent.

<u>Proof</u> : For $1 \leq k \leq 3$, the proof of the $\underline{V}_T$-unisolvence of Σ_T can be found in [12].

In [12] it has also been stated that for $k > 3$, there is <u>no</u> proof for the $\underline{V}_T$-unisolvence of Σ_T. But recently the following result has been obtained :

<u>Lemma</u> (5.2) : For $k \geq 6$, Σ_T defined by (5.14) is <u>not</u> $\underline{V}_T$-unisolvent.

<u>Proof</u> : It is sufficient to show that for $k \geq 6$, $\exists \; \Phi_h^* = (\phi_{hij}^*) \neq 0$ in $\underline{V}_T$ such that

$$\int_{T_i^*} M_n(\Phi_h^*) q \, ds = 0, \; q \in P_k(T_i^*), \; 1 \leq i \leq 3 \; ; \tag{5.16}$$

$$\int_{T_i^*} K_n(\Phi_h^*) q \, ds = 0, \; q \in P_{k-1}(T_i^*), \; 1 \leq i \leq 3 \; ; \tag{5.17}$$

$$\int_T \phi_{hij}^* \, b_{,ij} \, p \, dT = 0, \; b \in B_{k+1}(T), \; p \in P_1(T). \tag{5.18}$$

The proof is given in three steps.

Step 1 : For $k \geq 6$, the following auxiliary set Σ_T^* of linearly independent linear functionals on $\underline{V}_T$ defined by : $\forall \Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$,

$$\Sigma_T^* = \{ \int_{T_i^*} M_n(\Phi_h^*) q \, ds, \ q \in P_k(T_i^*), \ 1 \leq i \leq 3 \ ; \int_{T_i^*} K_n(\Phi_h^*) q \, ds, \ q \in P_{k-1}(T_i^*), \ 1 \leq i \leq 3 \ ; \int_T (\phi_{h11,1}^* + \phi_{h21,2}^*) b \, dT, \ b \in B_{k+2}(T), \int_T (\phi_{h12,1}^* + \phi_{h22,2}^*) b \, dT, \ b \in B_{k+2}(T) \} \tag{5.19}$$

is not $\underline{V}_T$-unisolvent.

Since for $k \geq 6$,
$$\begin{aligned} \mathrm{Card}(\Sigma_T^*) &= 3(k+1) + 3k + \frac{k(k+1)}{2} + \frac{k(k+1)}{2} \\ &= 3(k+1) + 3k + k(k+1) \\ &< 3(k+1) + 3k + 3(k-1)k/2 = \dim \underline{V}_T \ , \end{aligned} \tag{5.20}$$

Σ_T^* defined by (5.19) is not $\underline{V}_T$-unisolvent, and hence, for $k \geq 6$, there exists a nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ (i.e. $\exists \ \Phi_h^* = (\phi_{hij}^*) \neq 0$ in $\underline{V}_T$) such that

$$\int_{T_i^*} M_n(\Phi_h^*) q \, ds = 0, \ q \in P_k(T_i^*), \ 1 \leq i \leq 3, \tag{5.21}$$

$$\int_{T_i^*} K_n(\Phi_h^*) q \, ds = 0, \ q \in P_{k-1}(T_i^*), \ 1 \leq i \leq 3, \tag{5.22}$$

$$\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*) b \, dT = 0, \ b \in B_{k+2}(T), \tag{5.23}$$

$$\int_T (\phi_{h12,1}^* + \phi_{h22,2}^*) b \, dT = 0, \ b \in B_{k+2}(T). \tag{5.24}$$

Step 2 : For $k \geq 6$, for any non zero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24),

$$M_n(\Phi_h^*) = 0 \quad \text{on} \quad \partial T \ , \tag{5.25}$$

$$\phi_{hij,ij}^* = 0 \ \text{in} \ \ T. \tag{5.26}$$

Since $M_n(\Phi_h^*|_{T_i^*}) \in P_k(T_i^*)$, $1 \leq i \leq 3$, from (5.21), the result (5.25) immediately follows, if we choose $q = M_n(\Phi_h^*|_{T_i^*})$, $1 \leq i \leq 3$.

Now, we prove (5.26) $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T \Rightarrow (\phi_{h11,1}^* + \phi_{h21,2}^*) \in P_{k-1}(T)$.

Then, choosing $b = \lambda_1\lambda_2\lambda_3(\phi_{h11,1}^* + \phi_{h21,2}^*) \in B_{k+2}(T)$, we get from (5.23) :

$$\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*)^2 \lambda_1\lambda_2\lambda_3 \, dT = 0 \Longleftrightarrow \phi_{h11,1}^* + \phi_{h21,2}^* = 0 \text{ in } T, \tag{5.27}$$

since $\lambda_i > 0$ in $\mathring{T}$, $1 \le i \le 3$.

Similarly, we get from (5.24) : $\phi_{h12,1}^* + \phi_{h22,2}^* = 0$ in T. (5.28)

Now, differentiating both sides of (5.27) and (5.28) with respect to x_1 and x_2 respectively and then adding, we get :

$$\phi_{hij,ij}^* = 0 \quad \text{in} \quad T. \tag{5.29}$$

<u>Step 3</u> : <u>For</u> $k \ge 6$, <u>any nonzero</u> $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ <u>satisfying (5.21)-(5.24) will also satisfy (5.16)-(5.18), i.e. for</u> $k \ge 6$, Σ_T <u>defined by (5.14) is not V_T-unisolvent</u>.

For $k \ge 6$, $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24) satisfies (5.16) and (5.17). So, it remains to show that such a nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ will also satisfy (5.18). For this, we write the left hand side of (5.18) as follows :

For $b \in B_{k+1}(T)$, $\bar{\alpha}_i \in \mathbb{R}$, $i = 0,1,2$, $\int_T \phi_{hij}^* b_{,ij}(\bar{\alpha}_o + \bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2)dT$

$$= \bar{\alpha}_o \int_T \phi_{hij}^* b_{,ij} \, dT + \bar{\alpha}_1 \int_T \phi_{hij}^* b_{,ij} x_1 \, dT + \bar{\alpha}_2 \int_T \phi_{hij}^* b_{,ij} x_2 \, dT. \tag{5.30}$$

Now, for $k \ge 6$, if each of the three integrals on the right hand side of (5.30) vanishes, then for $k \ge 6$, any nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24) will also satisfy (5.18). Hence, it is sufficient to prove that for $k \ge 6$, for nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24), we have :

$$\int_T \phi_{hij}^* b_{,ij} \, dT = 0, \; b \in B_{k+1}(T), \tag{5.31}$$

$$\int_T \phi_{hij}^* b_{,ij} x_1 dT = 0, \; b \in B_{k+1}(T), \tag{5.32}$$

$$\int_T \phi_{hij}^* b_{,ij} x_2 dT = 0, \; b \in B_{k+1}(T). \tag{5.33}$$

From the Green's formula (Corollary (4.1)), we have : $\forall \Phi_h = (\phi_{hij})$ with $\phi_{hij} = \phi_{hji} \in P_m(T)$, $m \ge 1$, $\forall b \in B_{k+1}(T)$, $k \ge 2$,

$$\int_T \phi_{hij}\, b_{,ij}\, dT = \int_T \phi_{hij,ij}\, b\, dT + \int_{\partial T} M_n(\Phi_h)\frac{\partial b}{\partial n}\, ds. \tag{5.34}$$

First of all, we will prove (5.31). In fact, for $k \geq 6$, for any nonzero $\Phi_h^* = (\phi_{hij}^*) \in V_T$ satisfying (5.21)-(5.24), we have (5.25)-(5.26). Then, the equality (5.31) follows from (5.34) (5.25) and (5.26).

Now, we prove (5.32). Since $x_1\ \phi_{hij}^* = x_1\ \phi_{hji}^* \in P_m(T)$ $\forall i,\ j = 1,2,\ m = k+1$, from (5.34) we have : for $k \geq 6$,

$$\int_T (\phi_{hij}^*\ x_1) b_{,ij}\, dT = \int_T (x_1\ \phi_{hij}^*)_{,ij}\, b\, dT + \int_{\partial T} M_n((x_1\phi_{hij}^*))\ \frac{\partial b}{\partial n}\, ds.$$

Since for $k \geq 6$, for nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24), we have : $M_n((x_1\phi_{hij}^*)) = x_1\ \phi_{hij}^*\ n_i\ n_j = 0$ by virtue of (5.25). Hence, for $k \geq 6$ for nonzero $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ satisfying (5.21)-(5.24), we have from (5.26) and (5.23) :

$$\begin{aligned}\int_T (\phi_{hij}^*\ x_1) b_{,ij}\, dT &= \int_T [(x_1\phi_{h11}^*)_{,11} + (x_1\phi_{h12}^*)_{,12} + (x_1\phi_{h21}^*)_{,21} + (x_1\phi_{h22}^*)_{,22}] b\, dT \\ &= \int_T x_1\ \phi_{hij,ij}^*\, b\, dT + 2\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*)\, b\, dT = 0.\end{aligned}$$

Similarly, (5.33) is proved. Thus, we have proved that for $k \geq 6$, $\exists$ a <u>nonzero</u> $\Phi_h^* = (\phi_{hij}^*) \in \underline{V}_T$ such that (5.16)-(5.18) hold, i.e. for $k \geq 6$, Σ_T defined by (5.14) is <u>not</u> $\underline{V}_T$-unisolvent.

<u>Proposition (5.3)</u> : For $1 \leq k \leq 3$, the degrees of freedom $(\Sigma 1)$ for tensor-valued functions $\Phi_h = (\phi_{hij}) \in \underline{V}_h$ can be defined by the values of :

$$(\Sigma 1) : \int_{T_i^*} M_n(\Phi_h) q\, ds,\ q \in P_k(T_i^*),\ 1 \leq i \leq 3\ ;\ \int_{T_i^*} K_n(\Phi_h) q\, ds,\ q \in P_{k-1}(T_i^*),\ 1 \leq i \leq 3, \tag{5.35}$$

$$\int_T \phi_{hij}(\lambda_1\lambda_2\lambda_3 p)_{,ij}\, q\, dT,\ p \in P_{k-2}(T),\ q \in P_1(T).$$

<u>Proof</u> : The result follows from the Lemma (5.1).

Remark (5.3) : From the Lemma (5.2), it follows that for $k \geq 6$, $(\Sigma 1)$ defined in (5.35) do <u>not</u> define degrees of freedom for functions $\Phi_h \in \underline{V}_h$, but for $k = 4,5$ the problem is still open i.e. it is still not known whether $(\Sigma 1)$ in (5.35) define degrees of freedom for functions $\Phi_h \in \underline{V}_h$ or not.

Define

$$\underline{Z}(f) = \{\Phi : \Phi \in \underline{V},\ b(\Phi,\chi) = -\langle f,\chi\rangle_{0,\Omega} \quad \forall \chi \in W\} \ ;$$
$$\underline{Z} = Z(0) = \{\Phi : \Phi \in \underline{V},\ b(\Phi,\chi) = 0 \quad \forall \chi \in W\} \ ; \tag{5.36}$$

$$\underline{Z}_h(f) = \{\Phi_h : \Phi_h \in \underline{V}_h,\ b(\Phi_h,\chi_h) = -\langle f,\chi_h\rangle_{0,\Omega} \ \forall \chi_h \in W_h\} \ , \tag{5.37}$$

$$\underline{Z}_h = Z_h(0) = \{\Phi_h : \Phi_h \in \underline{V}_h,\ b(\Phi_h,\chi_h) = 0 \quad \forall \chi_h \in W_h\} \ , \tag{5.38}$$

$\underline{Z}$ and $\underline{Z}_h$ being subspaces of $\underline{V}$ and $\underline{V}_h$ $(\subset \underline{V})$ respectively.

We can rewrite (4.25) as follows :

$\forall \Phi \in \underline{V}$, $\forall \chi \in W$,

$$b(\Phi,\chi) = -\sum_{T \in T_h} \int_T \phi_{ij,ij}\ \chi \, dT - \sum_{a \in N_{oh}} B(a,\Phi)\ \chi(a), \tag{5.39}$$

where N_{oh} is the set of interior vertices of T_h in Ω .

Lemma (5.3) : Let $(\Sigma 1)$ be the degrees of freedom of $\Phi_h \in \underline{V}_h$ defined in (5.35). If $\Phi_h = (\phi_{hij}) \in \underline{Z}_h$, then

$$\phi_{hij,ij} = 0 \quad \text{in each } T \in T_h \tag{5.40}$$

$$B(a,\Phi_h) = 0 \quad \text{at each vertex } a \in N_{oh}, \tag{5.41}$$

where $B(a,\Phi_h)$ is defined in (5.39).

Proof : The following proof is given in [25].

Let $\Phi_h = (\phi_{hij}) \in \underline{Z}_h$. Then, $\forall \chi_h \in W_h$, $b(\Phi_h,\chi_h) = 0$

$$\Rightarrow -\sum_{T \in T_h} \int_T \phi_{hij,ij}\ \chi_h \, dT - \sum_{a \in N_{oh}} B(a,\Phi_h)\chi_h(a) = 0.$$

We prove first (5.40). For $k = 1$, the (5.40) is trivially satisfied. Now, assume $k \geq 2$. For any fixed $T \in T_h$, for any $\Phi_h = (\phi_{hij}) \in \underline{Z}_h$, choose χ_h^* as follows :

$$\chi_h^* = \begin{cases} \phi_{hij,ij}\ \lambda_1\lambda_2\lambda_3 & \text{in } \overset{\circ}{T} = \mathrm{int}(T) \\ 0 & \text{in } \Omega - \overset{\circ}{T} . \end{cases} \tag{5.42}$$

Then, $\chi_h^* \in W_h$ and $\forall \Phi_h \in \underline{Z}_h$,

$$b(\Phi_h,\chi_h^*) = - \int_T (\phi_{hij,ij})^2\ \lambda_1\lambda_2\lambda_3\ dT = 0 \Rightarrow \phi_{hij,ij} = 0 \text{ in each } T \in T_h \quad \forall \Phi_h \in \underline{Z}_h .$$

Now, we prove (5.41). Let $a \in N_{oh}$ be an interior vertex of T_h and $k \geq 1$. Then, $\exists$ at least one function $\tilde{\chi}_h \in W_h$ such that

$$\tilde{\chi}_h(x) = \begin{cases} B(a,\Phi_h) & \text{for } x = a \\ 0 & \text{for } x \in N_{oh} - \{a\} . \end{cases}$$

Then, $\forall \Phi_h \in \underline{Z}_h$, $b(\Phi_h,\tilde{\chi}_h) = -(B(a,\Phi_h))^2 = 0 \Rightarrow B(a,\Phi_h) = 0$ $\forall \Phi_h \in \underline{Z}_h$.

Although $\underline{Z}_h \not\subset \underline{Z}$ in general, but in this particular case of construction of $\underline{V}_h$, $\underline{Z}_h \subset \underline{Z}$. In fact, we have

Lemma (5.4) : Let $(\Sigma 1)$ be the degrees of freedom of $\Phi_h \in \underline{V}_h$ defined by (5.35). Then, the inclusion $\underline{Z}_h \subset \underline{Z}$ holds.

Proof : Let $\Phi_h \in \underline{Z}_h$. Then, from (5.40), (5.41), $b(\Phi_h,\chi) = 0$ $\forall \chi \in W \Rightarrow \Phi_h \in \underline{Z}$.

Now, we define the linear operator $\underline{\Pi}_h \in \mathcal{L}(\underline{V},\underline{V}_h)$ as follows [11] , [25] : $\forall \Phi \in \underline{V}$, $\underline{\Pi}_h\Phi \in \underline{V}_h$ such that

$$\int_{T^*} M_n(\Phi - \underline{\Pi}_h \Phi) q\, ds = 0,\ q \in P_k(T^*),\ T^* \in S_h,\ 1 \leq k \leq 3 \tag{5.43}$$

$$\int_{T^*} K_n(\Phi - \underline{\Pi}_h\Phi) q\, ds = 0,\ q \in P_{k-1}(T),\ T^* \in S_h,\ 1 \leq k \leq 3 \tag{5.44}$$

$$\int_T (\phi_{ij} - (\underline{\Pi}_h\Phi)_{ij})(\lambda_1\lambda_2\lambda_3\ p)_{,ij}\ q\, dT = 0,\ p \in P_{k-2}(T),\ q \in P_1(T),\quad 1 \leq i,j \leq 2,\ 2 \leq k \leq 3 \tag{5.45}$$

Lemma (5.5) [25] : Let $\underline{\Pi}_h : \underline{V} \rightarrow \underline{V}_h$ be the linear operator defined by (5.43)-(5.45). Then, $\forall \Phi \in \underline{V}$, $b(\Phi - \underline{\Pi}_h\Phi,\chi_h) = 0$ $\forall \chi_h \in W_h$. (5.46)

Proof : From (4.14) and (4.25), $\forall \chi_h \in W_h$, $\forall \Phi \in \underline{V}$, $b(\Phi - \underline{\Pi}_h \Phi, \chi_h) =$

$$- \sum_{T \in T_h} \left[\int_T (\Phi - \underline{\Pi}_h \Phi)_{ij} \; \chi_{h,ij} \; dT - \sum_{i=1}^{3} \int_{T_i^*} \{ M_n(\Phi - \underline{\Pi}_h \Phi) \frac{\partial \chi_h}{\partial n} - K_n(\Phi - \underline{\Pi}_h \Phi) \chi_h \} ds \right]$$

$$= \sum_{T^* \in S_h} \int_{T^*} K_n(\Phi - \underline{\Pi}_h \Phi) \chi_h \; ds = 0, \text{ since}$$

$$\chi_h|_T \in P^*_{k+1}(T) \subset P_{k+1}(T) \;\; \forall T \in T_h \;\Rightarrow\; \frac{\partial \chi_h}{\partial n}\Big|_{T_i^*} \in P_k(T_i^*), \; 1 \le i \le 3,$$

$$(\chi_h|_T)_{,ij} = (\lambda_1 \lambda_2 \lambda_3 p)_{,ij} \text{ with } p \in P_{k-2}(T)$$

$$\Rightarrow \int_T (\Phi - \underline{\Pi}_h \Phi)_{ij} \; \chi_{h,ij} \; dT = 0 \quad \forall \Phi \in \underline{V}, \forall T \in T_h \;;$$

$$\int_{T_i^*} M_n(\Phi - \underline{\Pi}_h \Phi) \frac{\partial \chi_h}{\partial n} ds = 0 \quad \forall \Phi \in \underline{V}, \forall T \in T_h \;;$$

and last integral vanishes by virtue of the continuity of $K_n(\Phi - \underline{\Pi}_h \Phi)$ at the interelement boundaries of T_h, continuity of χ_h and $\chi_h|_\Gamma = 0$.

Lemma (5.6) [11],[25]: Let $\{T_h\}$ be a regular family [12] of triangulations of $\bar{\Omega}$ and $\underline{\Pi}_h \in \mathcal{L}(\underline{V}, \underline{V}_h)$ be defined by (5.43)-(5.45). Then, $\exists\, C^* > 0$, independent of h, such that

$$\| \underline{\Pi}_h \Phi \|_{\underline{V}} \le C^* \| \Phi \|_{\underline{V}} \qquad \forall \Phi \in \underline{V}. \tag{5.47}$$

Moreover, $\forall \Phi \in (H^{k+1}(\Omega))^4 \cap \underline{V}$, $\exists\, C_1^* > 0$ independent of h, such that for $1 \le k \le 3$

$$\| \Phi - \underline{\Pi}_h \Phi \|_{\ell,\Omega} \le C_1^* \; h^{k+1-\ell} \; |\Phi|_{k+1,\Omega}, \; 0 \le \ell \le k+1. \tag{5.48}$$

Proposition (5.4) : For regular family $\{T_h\}$ of triangulations of $\bar{\Omega}$ and for degrees of freedom ($\Sigma 1$) defined in (5.35), the assumption (A3) holds.

Proof : Corresponding to ($\Sigma 1$) defined in (5.35) let $\underline{\Pi}_h \in \mathcal{L}(\underline{V}, \underline{V}_h)$ be defined by (5.43)-(5.45). Then, from (4.29) and (5.47), we have $\forall \chi_h \in W_h$,

$$\sup_{\Phi_h \in \underline{V}_h} \frac{|b(\Phi_h, \chi_h)|}{\|\Phi_h\|_{\underline{V}}} \ge \sup_{\Phi \in \underline{V}} \frac{|b(\underline{\Pi}_h \Phi, \chi_h)|}{\|\underline{\Pi}_h \Phi\|_{\underline{V}}}$$

$$= \sup_{\Phi \in \underline{V}} \frac{|b(\Phi, \chi_h)|}{\|\Phi\|_{\underline{V}}} \frac{\|\Phi\|_{\underline{V}}}{\|\underline{\Pi}_h \Phi\|_{\underline{V}}} \ge \frac{\beta}{C^*} \; \|\chi_h\|_{0,\Omega},$$

from which the result follows with $\beta_1 = \beta/C^* > 0$.

Corresponding to $(\Sigma 1)$ we constructed $\underline{\Pi}_h \in \mathcal{L}(\underline{V},\underline{V}_h)$ to prove that (A3) holds, which, in turn, assures the existence and uniqueness of the solution of the equilibrium finite element problem (Q_h).
But, conversely, we have

<u>Proposition (5.5)</u> [13] : If the assumption (A3) holds, then $\exists$ an operator $\underset{\sim}{\Pi}_h \in \mathcal{L}(\underline{V},\underline{V}_h)$ satisfying :

$b(\Phi-\underset{\sim}{\Pi}_h\Phi,\chi_h) = 0 \quad \forall\, \Phi \in \underline{V},\ \forall \chi_h \in W_h,$

$\|\underset{\sim}{\Pi}_h\Phi\|_{\underline{V}} \le \frac{m^*}{\beta_1} \|\Phi\|_{\underline{V}}$, m^* and β_1 being positive constants in (4.27) and (A3) respectively.

Moreover, we have the following interesting result which we do not need for our subsequent analysis :

<u>Proposition (5.6)</u> : Let $\{T_h\}$ be a regular family of triangulations of $\bar{\Omega}$. Let $(\Sigma 1)$ be the degrees of freedom of $\Phi_h \in \underline{V}_h$ defined by (5.35). Then, $\exists$ an operator $\Pi_h^* \in \mathcal{L}(W,W_h)$ such that

$$b(\Phi_h,\chi-\Pi_h^*\chi) = 0 \quad \forall\chi \in W, \tag{5.49}$$

$$\|\Pi_h\chi\|_W \le C_2^* \|\chi\|_W \quad \text{for some } C_2^* > 0,\ \forall\chi \in W. \tag{5.50}$$

<u>Proof</u> : Let $(\Sigma 1)$ be the degrees of freedom of $\Phi_h \in \underline{V}_h$ corresponding to a regular family $\{T_h\}$ of triangulations. Then, from the Lemma (5.4), we have $\underline{Z}_h \subset \underline{Z}$, and the result follows from the Proposition 3 in [13].

6. <u>ERROR ESTIMATES</u> : Define

$$X_h = \{\chi_h : \chi_h \in C^0(\bar{\Omega}),\ \ \chi_h|_T \in P_1(T)\ \forall T \in T_h\} \subset W_h. \tag{6.1}$$

Then, for $k = 1$, we have from (5.3) and (5.6), $X_h = W_h$. (6.2).
Let $g_h : W \to X_h$ be defined by :

$$\forall\chi \in W \subset C^0(\bar{\Omega}),\ g_h\chi \in X_h \text{ and } (g_h\chi)(a_{i,T}) = \chi(a_{i,T}),\ 1 \le i \le 3,\ \forall T \in T_h. \tag{6.3}$$

Remark (6.1) : $\forall\chi \in W$, $g_h\chi$ is the standard Lagrange interpolant of χ at the vertices of triangles $T \in T_h$.

Proposition (6.1) : $\forall\Phi_h \in \underline{V}_h$, $\forall\chi \in W$,
$$|b(\Phi_h,\chi - g_h\chi)| \leq \|\Phi_h\|_{\underline{V}} \; \|\chi - g_h\chi\|_{0,\Omega}, \tag{6.4}$$

where $g_h \in \mathcal{L}(W,X_h)$ is defined by (6.3).

Proof : Since $(\chi - g_h\chi)(a_{i,T}) = 0 \;\; \forall i = 1,2,3, \;\; \forall T \in T_h$, the result follows from the application of the Cauchy-Schwarz inequality to (4.25).

Theorem (6.1) : For $0 < h < 1$, let $\{T_h\}$ be a regular family of triangulations of $\bar{\Omega}$. Then, $\exists$ constants $C_1, C_2 > 0$, independent of h, such that

$$\|\Psi - \Psi_h\|_{0,\Omega} \leq C_1\|\Psi - \underline{\Pi}_h\Psi\|_{0,\Omega} \;; \tag{6.5}$$

$$\|\lambda - \lambda_h\|_{0,\Omega} \leq C_2(\|\lambda - g_h\lambda\|_{0,\Omega} + \|\Psi - \Psi_h\|_{0,\Omega}), \tag{6.6}$$

where $(\Psi,\lambda) \in \underline{V} \times W$, $(\Psi_h,\lambda_h) \in \underline{V}_h \times W_h$ are the solutions of the problems (Q) and (Q_h) respectively,

$\underline{\Pi}_h \in \mathcal{L}(\underline{V},\underline{V}_h)$, $g_h \in \mathcal{L}(W,X_h)$ are defined by (5.22)-(5.24) and (6.1)-(6.3) respectively.

Proof : From the Lemma (5.3), $\forall\Phi \in \underline{V}$, $b(\Phi - \underline{\Pi}_h\Phi,\chi_h) = 0 \;\; \forall\chi_h \in W_h$.

Then, $b(\Psi - \underline{\Pi}_h\Psi,\chi_h) = 0 \;\forall\chi_h \in W_h \Rightarrow b(\underline{\Pi}_h\Psi,\chi_h) = b(\Psi,\chi_h) = -<f,\chi_h>$
$\forall\chi_h \in W_h \Rightarrow \underline{\Pi}_h\Psi \in \underline{Z}_h(f)$ (by (5.16)). (6.7)

Since $\Psi_h \in \underline{Z}_h(f)$, $\forall\Phi_h \in \underline{Z}_h(f)$, $\Psi_h - \Phi_h \in \underline{Z}_h \Rightarrow \Psi_h - \Phi_h \in \underline{Z}$ by virtue of the inçlusion $\underline{Z}_h \subset \underline{Z}$ proved in the Lemma (5.2). Then, $b(\Psi_h - \Phi_h, \chi - \chi_h) = 0 \;\; \forall\,\Phi_h \in \underline{Z}_h(f)$, $\forall\chi \in W$, $\forall\chi_h \in W_h$.

Hence, $b(\Psi_h - \Phi_h, \lambda - \lambda_h) = 0 \;\; \forall\Phi_h \in Z_h(f) \Rightarrow b(\Psi_h - \underline{\Pi}_h\Psi, \lambda - \lambda_h) = 0$ (6.8)
by (6.7).
Again, from (4.33), (5.8) and (6.8) we have $A(\Psi - \Psi_h, \Psi_h - \underline{\Pi}_h\Psi) +$
$b(\Psi_h - \underline{\Pi}_h\Psi, \lambda - \lambda_h) = A(\Psi - \Psi_h, \Psi_h - \underline{\Pi}_h\Psi) = 0$. (6.9)

Then, from (4.26), (4.28) and (6.9), we have

$$\alpha_2 \; \| \Psi_h - \underline{\Pi}_h \Psi \|^2_{0,\Omega} \leq A(\Psi_h - \underline{\Pi}_h\Psi, \Psi_h - \underline{\Pi}_h\Psi)$$

$$= A(\Psi - \underline{\Pi}_h\Psi, \Psi_h - \underline{\Pi}_h\Psi) - A(\Psi-\Psi_h, \Psi_h - \underline{\Pi}_h\Psi)$$

$$= A(\Psi - \underline{\Pi}_h\Psi, \Psi_h - \underline{\Pi}_h\Psi) \leq M \| \Psi - \underline{\Pi}_h\Psi \|_{0,\Omega} \| \Psi_h - \underline{\Pi}_h\Psi \|_{0,\Omega}$$

$$\Rightarrow \| \Psi_h - \underline{\Pi}_h\Psi \|_{0,\Omega} \leq \frac{M}{\alpha_2} \| \Psi - \underline{\Pi}_h\Psi \|_{0,\Omega} . \tag{6.10}$$

Now, from (6.10) and the inequality : $\| \Psi - \Psi_h \|_{0,\Omega} \leq \| \Psi - \underline{\Pi}_h\Psi \|_{0,\Omega} + \| \Psi_h - \underline{\Pi}_h\Psi \|_{0,\Omega}$, the result (6.5) follows with $C_1 = (1+M/\alpha_2) > 0$.

Now, we prove (6.6). From (4.33) and (5.8) we have : $\forall \Phi_h \in \underline{V}_h$

$b(\Phi_h, \lambda_h - \lambda) = A(\Psi-\Psi_h, \Phi_h)$

$$\Rightarrow b(\Phi_h, \lambda_h - g_h\lambda) + b(\Phi_h, g_h\lambda - \lambda) = A(\Psi-\Psi_h, \Phi_h) \;\; \forall \Phi_h \in \underline{V}_h , \tag{6.11}$$

where $g_h \in \mathcal{L}(W, X_h)$ is defined by (6.1)-(6.3).

Then, since for $\underline{\Pi}_h \in \mathcal{L}(\underline{V}, \underline{V}_h)$ defined by (5.22)-(5.24) corresponding to (Σ1) in (5.12)-(5.14), the (A3) holds, we have from (6.11) :

$$\beta_1 \| \lambda_h - g_h\lambda \|_{0,\Omega} \leq \sup_{\Phi_h \in \underline{V}_h} \frac{|b(\Phi_h, \lambda_h - g_h\lambda)|}{\|\Phi_h\|_{\underline{V}}}$$

$$\leq \sup_{\Phi_h \in \underline{V}_h} \frac{|A(\Psi-\Psi_h, \Phi_h)|}{\|\Phi_h\|_{\underline{V}}} + \sup_{\Phi_h \in \underline{V}_h} \frac{|b(\Phi_h, \lambda - g_h\lambda)|}{\|\Phi_h\|_{\underline{V}}}$$

$$\leq M \|\Psi - \Psi_h\|_{0,\Omega} + \|\lambda - g_h\lambda\|_{0,\Omega} \text{ (by virtue of (6.4))}$$

$$\Rightarrow \|\lambda_h - g_h\lambda\|_{0,\Omega} \leq (M/\beta_1) \| \Psi-\Psi_h \|_{0,\Omega} + \frac{1}{\beta_1} \|\lambda - g_h\lambda\|_{0,\Omega} . \tag{6.12}$$

Now, from (6.12) and the inequality : $\| \lambda - \lambda_h \|_{0,\Omega} \leq \| \lambda - g_h\lambda \|_{0,\Omega} + \| \lambda_h - g_h\lambda \|_{0,\Omega}$, the result (6.6) follows with $C_2 = \max \{1 + \frac{1}{\beta_1} , M/\beta_1\}$.

The final result is given by :

Theorem (6.2) : For $0 < h < 1$, and let $\underline{V}_h$, W_h be the finite dimensional vector spaces defined by (5.5)-(5.6) corresponding to a regular family $\{T_h\}$ of triangulations of $\bar{\Omega}$ and the degrees of freedom ($\Sigma 1$) defined by (5.12)-(5.14).

If the solution $u \in H_0^2(\Omega)$ of the problem (P_G) belongs to $H^{k+3}(\Omega) \cap H_0^2(\Omega)$, $1 \leq k \leq 3$ such that $a_{ijml}\, u_{,ml}$ $H^{k+1}(\Omega)$ $\forall i,j = 1,2$, then $\exists$ constants C_3 and $C_4 > 0$, independent of h, such that

$$\|\Psi - \Psi_h\|_{0,\Omega} \leq C_3\, h^{k+1}\, |\Psi|_{k+1,\Omega}\ (1 \leq k \leq 3)\ ; \tag{6.13}$$

$$\|\lambda - \lambda_h\|_{0,\Omega} \leq C_4\, h^2 (|\lambda|_{2,\Omega} + |\Psi|_{k+1,\Omega})\ \ (1 \leq k \leq 3), \tag{6.14}$$

where $(\Psi,\lambda) \in \underline{V} \times W$ is the solution of the problem (Q) with $\lambda = u$, $\Psi = (\psi_{ij})$, $1 \leq i,j \leq 2$, $\psi_{ij} = a_{ijml}\, u_{,ml}$ $\forall i,j = 1,2$; $(\Psi_h,\lambda_h) \in \underline{V}_h \times W_h$ is the equilibrium finite element solution of (Q_h).

Proof For $1 \leq k \leq 3$ let $u \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$ be the solution of the problem (P_G) with $a_{ijml}\, u_{,ml} \in H^{k+1}(\Omega)$ $\forall i,j = 1,2$. Then, the solution $(\Psi,\lambda) \in \underline{V} \times W$ of the problem (Q) will have the regularity defined by :

$$\lambda = u \in H^{k+3}(\Omega) \cap H_0^2(\Omega),\ \Psi = (\psi_{ij}) \in (H^{k+1}(\Omega))^4\ ;\ \ 1 \leq k \leq 3, \tag{6.15}$$

since $\forall i,j = 1,2$, $\psi_{ij} = a_{ijml}\, \lambda_{,ml} = a_{ijml}\, u_{,ml} \in H^{k+1}(\Omega)$ by virtue of (4.36).

Then, from (5.48), we have : $\|\Psi - \Pi_h \Psi\|_{0,\Omega} \leq C_1^* h^{k+1} |\Psi|_{k+1,\Omega}$, $1 \leq k \leq 3$, (6.16)

and the result (6.13) follows from (6.5) with $C_3 = C_1\, C_1^* > 0$.

Now, we prove (6.14). Since $(g_h\lambda)|_T \in P_1(T)$ $\forall T \in T_h$, $\lambda \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$, $1 \leq k \leq 3$, we have the classical result [12] : $\|\lambda - g_h\lambda\|_{0,\Omega} \leq C_5\, h^2 |\lambda|_{2,\Omega}$, (6.17)

$C_5 > 0$.

Then, from (6.6), (6.16) and (6.13), the result (6.14) follows with $C_4 = \max\{C_2 C_5, C_2 C_3\} > 0$.

7. Numerical Experiments

For the purpose of computation it will be much more convenient [11] to relax the constraint of 'continuity' of M_n and K_n across interelement boundaries of T_h in the definition of the admissible space $\underline{V}_h$(5.5) of the equilibrium finite element problem (Q_h) (5.8)-(5.9) by introducing suitable Lagrange multipliers and to construct an associated new discrete problem (Q_h^*) equivalent to (Q_h) in certain sense. Then, computations for numerical experiments on problems of interest can be easily carried out using this new scheme (Q_h^*) instead of the scheme (Q_h). Hence, before dealing with the numerical experiments involving computations, we are going to construct this new discrete problem (Q_h^*) and establish the relation between the problems (Q_h^*) and (Q_h). For this, corresponding to T_h, we introduce the finite dimensional spaces M_{1h}, M_{2h} as follows: For fixed $k \geq 1$,

$$M_{1h} = \{\mu_{1h} : \mu_{1h}|_L \in P_k(L) \; \forall L \in S_h^o, \; \mu_{1h}|_\Gamma = 0\},$$

$$M_{2h} = \{\mu_{2h} : \mu_{2h}|_L \in P_{k+1}(L) \; \forall L \in S_h^o, \; \mu_{2h}|_\Gamma = 0\}, \qquad (7.1)$$

where S_h^o is the set of interior sides L of T_h, i.e.
$L \in S_h^o \Rightarrow L \not\subset \Gamma$, $L \in S_h$.

Now, corresponding to T_h, we define the finite dimensional product space $\underline{M}_h$ of Lagrange multipliers and $\underline{V}_h^*$ as follows:

$$\underline{M}_h = \{\underline{\mu}_h : \underline{\mu}_h = (\mu_{1h}, \mu_{2h}) \text{ with } \mu_{1h} \in M_{1h}, \; \mu_{2h} \in M_{2h}\},$$

$$\underline{V}_h^* = \{ \underline{\Phi}_h^* : \underline{\Phi}_h^* = (\phi_{hij}^*)_{1 \leq i,j \leq 2} \, , \, \phi_{h12}^* = \phi_{h21}^* \, ,$$

$$\phi_{hij}^* \big|_T \in P_k(T) \; \forall T \in T_h \} \tag{7.2}$$

with $\underline{V}_h \subsetneqq \underline{V}_h^*$.

Define the bilinear form $c(\cdot,\cdot)$: $\underline{V}_h^* \times \underline{M}_h \to R$ as follows:

$$\forall \underline{\Phi}_h^* = (\phi_{hij}^*) \in \underline{V}_h^* \, , \quad \underline{\mu}_h = (\mu_{1h}, \mu_{2h}) \in \underline{M}_h \, ,$$

$$c(\underline{\Phi}_h^*, \underline{\mu}_h) = \sum_{T \in T_h} \int_{\partial T} -[\mu_{1h} M_n(\underline{\Phi}_h^*) \vec{n}_T \cdot \vec{n}_L - \mu_{2h} K_n(\underline{\Phi}_h^*)] ds, \tag{7.3}$$

where $\vec{n}_T$ is the unit normal exterior to T ; $\vec{n}_L$ is one of the two unit vectors normal to $L \in S_h$, $L \subset \partial T$;

$\vec{n}_T \cdot \vec{n}_L = \pm 1$; for $L \in S_h^o \subset S_h$ with $L = T_i \cap T_j (i \neq j)$

$\vec{n}_{T_i} \cdot \vec{n}_L = -(\vec{n}_{T_j} \cdot \vec{n}_L)$, $\vec{n}_{T_i}$ and $\vec{n}_{T_j}$ being unit normals exterior to T_i and T_j respectively.

Then, the new discrete problem (Q_h^*) is defined as follows:

Find $(\Psi_h^*, \lambda_h^*, \underline{\mu}_h^*) \in \underline{V}_h^* \times W_h \times \underline{M}_h$ such that

$$A(\Psi_h^*, \underline{\Phi}_h^*) + b(\underline{\Phi}_h^*, \lambda_h^*) + c(\underline{\Phi}_h^*, \underline{\mu}_h^*) = 0 \quad \forall \underline{\Phi}_h^* \in \underline{V}_h^*, \tag{7.4}$$

$$(Q_h^*): \quad b(\Psi_h^*, \chi_h) = - \langle f, \chi_h \rangle_{o,\Omega} \; \forall \chi_h \in W_h, \tag{7.5}$$

$$c(\Psi_h^*, \underline{\mu}_h) = 0 \quad \forall \underline{\mu}_h \in \underline{M}_h \, , \tag{7.6}$$

where the bilinear forms $A(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ are defined by (4.24),(4.25) and (7.3) respectively.

Lemma (7.1): (i) $\underline{\Phi}_h^* \in \underline{V}_h^*$, $c(\underline{\Phi}_h^*, \underline{\mu}_h) = 0 \ \forall \underline{\mu}_h \in \underline{M}_h \iff \underline{\Phi}_h^* \in \underline{V}_h$, (7.7)

where $\underline{V}_h$ is defined by (5.5),

(ii) $c(\underline{\Phi}_h^*, \underline{\mu}_h) = 0 \ \forall \underline{\Phi}_h^* \in \underline{V}_h^* \Rightarrow \underline{\mu}_h = 0 \in \underline{M}_h$ (7.8)

Theorem (7.1): Let (A3) hold. Then, if $(\Psi_h^*, \lambda_h^*, \underline{\mu}_h^*) \in \underline{V}_h^* \times W_h \times \underline{M}_h$ be a solution of the problem (Q_h^*), $\Psi_h^* = \Psi_h \in \underline{V}_h$, $\lambda_h^* = \lambda_h \in W_h$ such that $(\Psi_h^*, \lambda_h^*) \in \underline{V}_h \times W_h$ is the unique solution of (Q_h).

Proof: Let $(\Psi_h^*, \lambda_h^*, \underline{\mu}_h^*) \in \underline{V}_h^* \times W_h \times \underline{M}_h$ be a solution of (Q_h^*). Then, (7.4)-(7.6) hold. From (7.6) and (7.7), $\Psi_h^* \in \underline{V}_h$.

But from (7.7), $\underline{\Phi}_h \in \underline{V}_h \Rightarrow c(\underline{\Phi}_h, \underline{\mu}_h) = 0$ $\forall \underline{\mu}_h \in \underline{M}_h$. Consequently, $\forall \underline{\Phi}_h \in \underline{V}_h$, $A(\Psi_h^*, \underline{\Phi}_h) + b(\underline{\Phi}_h, \lambda_h^*) = 0$,

$\forall \chi_h \in W_h$, $b(\Psi_h^*, \chi_h) = -\langle f, \chi_h \rangle_{0,\Omega}$,

i.e. $(\Psi_h^*, \lambda_h^*) \in \underline{V}_h \times W_h$ is a solution of (Q_h) and its uniqueness follows from the Theorem (5.1) with $\Psi_h^* = \Psi_h \in \underline{V}_h$, $\lambda_h^* = \lambda_h \in W_h$.

Theorem (7.2): Let (A3) hold. Then, the problem (Q_h^*) has a unique solution.

Proof: Since (Q_h^*) is a finite dimensional problem, for the proof of the existence and uniqueness of solution of (Q_h^*), it is sufficient to prove that the corresponding homogeneous problem:

$$A(\Psi_h^*, \underline{\Phi}_h^*) + b(\underline{\Phi}_h^*, \lambda_h^*) + c(\underline{\Phi}_h^*, \underline{\mu}_h^*) = 0 \quad \forall \underline{\Phi}_h^* \in \underline{V}_h^* \tag{7.9}$$

$$b(\Psi_h^*, \chi_h) = 0 \quad \forall \chi_h \in W_h , \tag{7.10}$$

$$c(\Psi_h^*, \underline{\mu}_h) = 0 \quad \forall \underline{\mu}_h \in \underline{M}_h , \tag{7.11}$$

has only trivial solution $\Psi_h^* = 0 \in \underline{V}_h^*$, $\lambda_h^* = 0 \in W_h$, $\underline{\mu}_h^* = 0 \in \underline{M}_h$. In fact, from (7.7) and (7.11), $\Psi_h^* \in \underline{V}_h$ and

$c(\underline{\Phi}_h, \underline{\mu}_h) = 0 \;\forall \underline{\Phi}_h \in \underline{V}_h \subset \underline{V}_h^*$. Then, for $\Psi_h^* \in \underline{V}_h$,

$$A(\Psi_h^*, \underline{\Phi}_h) + b(\underline{\Phi}_h, \lambda_h^*) = 0 \;\forall \underline{\Phi}_h \in \underline{V}_h, \tag{7.12}$$

$$b(\Psi_h^*, \chi_h) = 0 \;\;\forall \chi_h \in W_h \tag{7.13}$$

$\Rightarrow \Psi_h^* = 0 \in \underline{V}_h, \lambda_h^* = 0 \in W_h$ is the unique (trivial) solution of (7.12)-(7.13) by the Theorem (5.1). Hence, from (7.9), $c(\underline{\Phi}_h^*, \underline{\mu}_h^*) = 0 \;\forall \underline{\Phi}_h^* \in \underline{V}_h^* \Rightarrow \underline{\mu}_h^* = 0 \in \underline{M}_h$ by virtue of (7.8).

Equivalence of (Q_h^*) and (Q_h)

From the Theorems (7.1) and (7.2) it is obvious that the two problems (Q_h^*) and (Q_h) are equivalent in the sense that the first two components Ψ_h^*, λ_h^* of the solution $(\Psi_h^*, \lambda_h^*, \underline{\mu}_h^*) \in \underline{V}_h^* \times W_h \times \underline{M}_h$ of the problem (Q_h^*) define the unique solution $(\Psi_h^*, \lambda_h^*) \in \underline{V}_h \times W_h$ of the problem (Q_h). Hence, instead of the scheme (Q_h), the new scheme (Q_h^*) can be used and will be much more convenient for numerical computations, since it will be much easier

(i) to construct basis functions in $\underline{V}_h^*$ than in $\underline{V}_h$,

(ii) to develop more efficient computer codes for the solution of algebraic system of equations of the scheme (Q_h^*) than of the scheme (Q_h).

Both (i) and (ii) result from the relaxation of the continuity requirements of the space $\underline{V}_h$ (5.5).

Reduction of (Q_h^*) into Matrix Form:

For the computational purpose, it is necessary to reduce the problem (Q_h^*) to matrix form, and the procedure for such a reduction will be outlined here. Let N1, N2, N3 be the dimensions and $\{\underline{\Phi}^i\}_{i=1}^{N1}$, $\{\chi^j\}_{j=1}^{N2}$, $\{\underline{\mu}^\ell\}_{\ell=1}^{N3}$ be the bases of spaces $\underline{V}_h^*, W_h, \underline{M}_h$ respectively. Then, the components of the solution $(\Psi_h^*, \lambda_h^*, \underline{\mu}_h^*) \in \underline{V}_h^* \times W_h \times \underline{M}_h$ can be represented as follows:

$$\Psi_h^* = \sum_{i=1}^{N1} \alpha_i \underline{\Phi}^i, \; \lambda_h^* = \sum_{j=1}^{N2} \beta_j \chi^j, \; \underline{\mu}_h^* = \sum_{\ell=1}^{N3} \gamma_\ell \underline{\mu}^\ell, \tag{7.14}$$

where $(\alpha_1,\alpha_2,\dots,\alpha_{N1})\in R^{N1}$, $(\beta_1,\beta_2,\dots,\beta_{N2})\in R^{N2}$, $(\gamma_1,\gamma_2,\dots,\gamma_{N3})\in R^{N3}$.

Now, define the matrices $[A],[B],[C]$ and the vector $\underline{F}$ as follows:

$$[A] = (a_{mn})_{N1\times N1} \text{ with } a_{mn} = A(\underline{\Phi}^m,\underline{\Phi}^n),\ 1\leq m,\ n\leq N1,$$

$$[B] = (b_{ij})_{N1\times N2} \text{ with } b_{ij} = b(\underline{\Phi}^i,\chi^j),\ 1\leq i\leq N1,\ 1\leq j\leq N2,$$

$$[C] = (c_{i\ell})_{N1\times N3} \text{ with } C_{i\ell} = c(\underline{\Phi}^i,\underline{\mu}^\ell),\ 1\leq i\leq N1,\ 1\leq \ell\leq N3, \qquad (7.15)$$

$$\underline{F} = (F_1,F_2,\dots,F_{N2})^T \text{ with } F_j = -\langle f,\chi^j\rangle_{o,\Omega},1\leq j\leq N2,$$

where $A(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ are defined by (4.24), (4.25),(7.3) respectively.

Then, the problem (Q_h^*) can be written in the matrix form as follows:
Find $(\underline{\alpha},\underline{\beta},\underline{\gamma})\in R^{N1+N2+N3}$ such that

$$\begin{pmatrix} [A] & [B] & [C] \\ [B]^t & [0] & [0] \\ [C]^t & [0] & [0] \end{pmatrix} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{F} \\ \underline{0} \end{pmatrix}, \qquad (7.16)$$

where $[A],[B]$, $[C]$ and $\underline{F}$ are defined by (7.15), $[B]^t$ (resp $[C]^t$) is the transpose of $[B]$ (resp. $[C]$), $\underline{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_{N1})^t$, $\underline{\beta} = (\beta_1,\beta_2,\dots,\beta_{N2})^t$, $\underline{\gamma} = (\gamma_1,\gamma_2,\dots,\gamma_{N3})^t$. The coefficient matrix in (7.16) is symmetric, but not positive definite. Hence, for the solution of (7.16), the Cholesky or modified Cholesky method of decomposition or the standard Frontal method can not be directly applied in general. But the matrix $[A]$ is symmetric and positive-definite. By assuming basis functions of $\underline{V}_h^*$ independently in each triangle $T\in T_h$ as shown in the sequel, $[A]$ can be given a block diagonal structure, i.e.

$$[A] = \begin{pmatrix} [A_{T_1}] & & & 0 \\ & [A_{T_2}] & & \\ & & \ddots & \\ 0 & & & [A_{T_{NT}}] \end{pmatrix} \qquad (7.17)$$

where the triangles of T_h have been globally ordered by numbering them as $T_1,T_2,\dots,T_{NT}$ with $\bigcup_{i=1}^{NT} T_i = \bar{\Omega}$, $T_i \neq T_j$ for $i\neq j$, NT being the total number of triangles in T_h, $[A_{T_i}]$ is the contribution only of the i-th triangle T_i, $1\leq i\leq NT$.

Hence, $[A_{T_i}]$, $1 \leq i \leq NT$, can be inverted immediately after its construction at the i-th element level, $[A_{T_i}]$ being symmetric, positive-definite and of order $3(k+2)(k+1)/2$. Thus

$$[A]^{-1} = \left\lceil\ [A_{T_1}]^{-1} \quad [A_{T_2}]^{-1} \quad \cdots \quad [A_{T_{NT}}]^{-1} \right\rfloor ,$$

where only the block diagonal matrices of $[A]^{-1}$ are shown, can be easily constructed with the help of $[A_{T_i}]^{-1}$, $1 \leq i \leq NT$. Then, from the first equation in (7.16), we get

$$\underline{\alpha} = -[A]^{-1}\,[B]\underline{\beta} - [A]^{-1}\,[C]\underline{\gamma} , \tag{7.18}$$

and for $\underline{\beta}$ and $\underline{\gamma}$, we have

$$[K]\begin{pmatrix} \underline{\beta} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{F} \\ \underline{O} \end{pmatrix}, \tag{7.19}$$

where

$$[K] = \begin{pmatrix} [B]^t[A]^{-1}[B] & [B]^t[A]^{-1}\,[C] \\ [C]^t[A]^{-1}[B] & [C]^t[A]^{-1}\,[C] \end{pmatrix} \tag{7.20}$$

is symmetric and positive-definite.
Now, (7.19) can be solved easily by the (modified) method of Cholesky decomposition or the Frontal method for $\underline{\beta}$, $\underline{\gamma}$. Finally, $\underline{\alpha}$ is determined from (7.18).

Remark (7.1): Preconditioned Conjugate gradient methods or other iteration techniques may be used for the solution of (7.16). Then, the inversion of $[A_{T_i}]$, $1 \leq i \leq NT$ at the element level will not be necessary. There are other techniques too for solving (7.16). But in the numerical experiments to be dealt with here, the above-mentioned procedure (7.17)-(7.20) will be followed.

Construction of Basis Functions:

Now, we shall indicate briefly the procedures for construction of basis functions $\{\underline{\Phi}^i\}_{i=1}^{N1}$, $\{\chi^j\}_{j=1}^{N2}$, $\{\underline{\mu}^\ell\}_{\ell=1}^{N3}$ in V_h^*, W_h, $\underline{M}_h$ introduced in (7.14) so that the standard finite element computational procedures can be conveniently carried out on computers.

i) Basis Functions in $\underline{V}_h^*$: Since tensor-valued functions in $\underline{V}_h^*$ are not required to satisfy any continuity requirements at the inter-element boundaries or vertices of T_h, these can be assumed independently in each triangle $T \in T_h$ and consequently, the construction of basis functions in $\underline{V}_h^*$ gets very much simplified as shown below. Consider any triangle T_j, $1 \leq j \leq NT$. For fixed $k \geq 1$, let $\{\phi^i_{T_j}\}_{i=1}^{N}$, $N = \dim P_k(T_j) = (k+2)(k+1)/2$, be a basis in $P_k(T_j)$, $1 \leq j \leq NT$. Define for $1 \leq i \leq N$,

$$\phi^{N(j-1)+i} = \begin{cases} \phi^i_{T_j} & \text{in } T_j \quad , \quad 1 \leq j \leq NT \\ 0 & \text{outside } T_j \ . \end{cases} \tag{7.21}$$

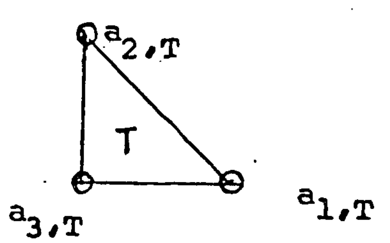
Now, define for $1 \leq j \leq N\mathrm{x}NT$,

$$\underline{\Phi}^{3(j-1)+1} = (\phi^j,0,0,0),\ \underline{\Phi}^{3(j-1)+2} = (0,\phi^j,\phi^j,0),\ \underline{\Phi}^{3(j-1)+3} = (0,0,0,\phi^j) \tag{7.22}$$

Then, $\{\underline{\Phi}^i\}_{i=1}^{N1}$ defines a basis in $\underline{V}_h^*$ with $N1 = 3N\mathrm{x}NT$.

Remark (7.2): For such a choice of basis functions in $\underline{V}_h^*$ as in (7.22), the matrix $[A]$ will have a block diagonal structure (7.17) allowing inversion of $[A_{T_i}]$, $1 \leq i \leq NT$ at the element level.

ii) Basis Functions in W_h: For $k = 1$, the degrees of freedom $\Sigma_{1,T}$ for the restrictions of functions $\chi_h \in W_h$ to each triangle $T \in T_h$ are given by: $\Sigma_{1,T} = \{\chi_h(a_{i,T})\}_{i=1}^{3}$, where $\{a_{i,T}\}_{i=1}^{3}$ are the three vertices of the triangle T, such that $\chi^i_T = \lambda_i$ (7.23) $(i = 1,2,3)$ are canonical basis functions in $P_2^*(T) = P_1(T)$ satisfying the property: $\chi^i_T(a_{j,T}) = \delta_{ij}$, $1 \leq i, j \leq 3$, $\{\lambda_i\}_{i=1}^{3}$ being the barycentric coordinates of a point in $T \in T_h$.



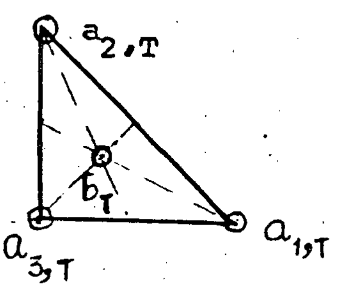
For $k = 1$, consider all the interior vertices and order them as $\{a_i\}_{i=1}^{N_{oh}}$ and introduce the degrees of freedom $\Sigma_1 = \{\chi_h(a_i)\}_{i=1}^{N_{oh}}$ and construct canonical basis functions $\{\chi^i\}_{i=1}^{N_{oh}}$ in W_h associated with each interior vertex a_i, $1 \leq i \leq N_{oh}$ such that

(i) $\quad \chi^i(a_j) = \delta_{ij}, \; 1 \leq i, \; j \leq N_{oh}$,

(ii) $\quad a_i = a_{j,T}$ for $T \in T_h \Rightarrow \chi^i|_T = \chi_T^j = \lambda_j \;, \; 1 \leq i \leq N_{oh},$
$1 \leq j \leq 3,$

$\dim W_h = N1 = N_{oh}$ for $k = 1$.

For $k = 2$, the degrees of freedom $\Sigma_{2,T}$ for the restrictions of functions $\chi_h \in W_h$ to each $T \in T_h$ are: $\Sigma_{2,T} = \left\{\chi_h(a_{i,T}), 1 \leq i \leq 3, \chi_h(b_T)\right\}$, where $\{a_{i,T}\}_{i=1}^{3}$ and $\{b_T\}$ are the vertices and barycentre of T respectively, such that



$$\chi_T^1 = \lambda_1(1-9\lambda_2\lambda_3), \; \chi_T^2 = \lambda_2(1-9\lambda_3\lambda_1), \; \chi_T^3 = \lambda_3(1-9\lambda_1\lambda_2), \quad (7.24)$$

$\chi_T^4 = 27\lambda_1\lambda_2\lambda_3$ are canonial basis functions in $P_3^*(T) = P_1(T) \oplus \lambda_1\lambda_2\lambda_3 \; P_o(T)$ satisfying the property:

$$\chi_T^i(a_{j,T}) = \delta_{ij} \;, \; 1 \leq i, \; j \leq 3, \; \chi_T^i(b_T) = 0, \; 1 \leq i \leq 3,$$

$$\chi_T^4(b_T) = 1, \; \chi_T^4(a_{i,T}) = 0, \quad 1 \leq i \leq 3.$$

Then, considering the set of $\{a_i\}_{i=1}^{N_{oh}}$ of interior vertices of T_h and the set of barycentres $\{b_i\}_{i=1}^{NT}$ of triangles $T_1, T_2, \ldots, T_{NT}$, and introducing the degrees of freedom $\Sigma_2 = \{\chi_h(a_i), \; 1 \leq i \leq N_{oh}, \chi_h(b_j), \; 1 \leq j \leq NT\}$, we can define canonical basis $\{\chi^i\}_{i=1}^{N_{oh}+NT}$ in W_h in a similar way as shown above for k=1 with $N1 = N_{oh} + NT = \dim W_h$.

iii) <u>Basis Functions in $\underline{M}_h$</u> : Consider the interior sides L of T_h, i.e. $L \in S_h^o$ with NS = number of interior sides of T_h = card(S_h^o). Then, for fixed $k \geq 1$, divide each $L \in S_h^o$ into (k+2) (resp. (k+1)) equal parts by means of (k+1) (resp. k) distinct points $\{q_{1L}^j\}_{j=1}^{k+1}$ (resp. $\{q_{2L}^j\}_{j=1}^{k}$) on each L to introduce the degrees of freedom Σ_L^1 (resp. Σ_L^2) for $\mu_{1h} \in P_k(L)$ (resp. $\mu_{2h} \in P_{k-1}(L)$) on each $L \in S_h^o$ as follows:

$$\Sigma_L^1 = \{\mu_{1h}(q_{1L}^j)\}_{j=1}^{k+1}, \quad \Sigma_L^2 = \{\mu_{2h}(q_{2L}^j)\}_{j=1}^{k}, \quad L \in S_h^o .$$

Hence, for fixed $k \geq 1$, $\Sigma^1 = \bigcup_{L \in S_h^o} \Sigma_L^1$, $\Sigma^2 = \bigcup_{L \in S_h^o} \Sigma_L^2$

will define the d.o.f. for functions in M_{1h} and M_{2h} associated with the interior nodes $\bigcup_{L \in S_h^o} \{q_{1L}^j\}_{j=1}^{k+1}$, $\bigcup_{L \in S_h^o} \{q_{2L}^j\}_{j=1}^{k}$ respectively, and

$$\mathrm{Card}(\Sigma^1) = \mathrm{Card}\Big(\bigcup_{L \in S_h^o} \{q_{1L}^j\}_{j=1}^{k+1}\Big) = NM1 = \dim M_{1h} ,$$

$$\mathrm{Card}(\Sigma^2) = \mathrm{Card}\Big(\bigcup_{L \in S_h^o} \{q_{2L}^j\}_{j=1}^{k}\Big) = NM2 = \dim M_{2h} .$$

Now, for fixed $k \geq 1$, $\forall$ fixed $L \in S_h^o$ consider the systems of functions $\{\mu_{1L}^j\}_{j=1}^{k+1}$, $\{\mu_{2L}^j\}_{j=1}^{k}$ belonging to M_{1h} and M_{2h} respectively and satisfying the following conditions:

$$\mu_{1L}^j\big|_L \in P_k(L), \quad \mu_{1L}^j\big|_{L^*} = 0 \quad \forall L^* \neq L,\ L^* \in S_h^o,\ 1 \leq j \leq k+1,$$

$$\mu_{1L}^j(q_{1L}^i) = \delta_{ij}, \quad 1 \leq i, j \leq k+1,$$

$$\mu_{2L}^j\big|_L \in P_{k-1}(L), \quad \mu_{2L}^j\big|_{L^*} = 0 \quad \forall L^* \neq L,\ L^* \in S_h^o,\ 1 \leq j \leq k,$$

$$\mu_{2L}^j(q_{2L}^i) = \delta_{ij}, \quad 1 \leq i, j \leq k.$$

Then $\forall L \in S_h^o$, $\forall k \geq 1$, Supp $\mu_{1L}^j = L$ $\forall j = 1,2,\ldots,k+1$, Supp $\mu_{2L}^j = L$ $\forall j = 1,2,\ldots,k$. Consider $\bigcup_{L \in S_h^o} \{\mu_{1L}^j\}_{j=1}^{k+1}$, $\bigcup_{L \in S_h^o} \{\mu_{2L}^j\}_{j=1}^{k}$.

Now, globally order the two sets $\bigcup_{L \in S_h^o} \{\mu_{1L}^j\}_{j=1}^{k+1}$ and $\bigcup_{L \in S_h^o} \{q_{1L}^j\}_{j=1}^{k+1}$ by renumbering as $\{\mu_1^j\}_{j=1}^{NM1}$, $\{q_1^j\}_{j=1}^{NM1}$ respectively such that $\mu_1^i(q_1^j) = \delta_{ij}$, $1 \leq i, j \leq NM1$.

Similarly, we get two sets globally ordered and renumbered as $\{\mu_2^i\}_{i=1}^{NM2}$, $\{q_2^i\}_{i=1}^{NM2}$ from $\bigcup_{L \in S_h^o} \{\mu_{2L}^i\}_{i=1}^{k}$, $\bigcup_{L \in S_h^o} \{q_{2L}^i\}_{i=1}^{k}$ respectively with the property $\mu_2^i(q_2^j) = \delta_{ij}$, $1 \leq i, j \leq NM2$. Then

$\{\mu_1^i\}_{i=1}^{NM1}$, $\{\mu_2^j\}_{j=1}^{NM2}$ define canonical bases in M_{1h}, M_{2h} with respect to the degrees of freedom Σ^1 and Σ^2 respectively,

Supp $\mu_1^i = L \in S_h^o$ with $q_1^i \in L$, $1 \leq i \leq NM1$,

Supp $\mu_2^j = L \in S_h^o$ with $q_2^j \in L$, $1 \leq j \leq NM2$.

and $\underline{\mu}^i = (\mu_1^i, 0)$, $1 \leq i \leq NM1$,

$$\underline{\mu}^{i+NM1} = (0, \mu_2^i), 1 \leq i \leq NM2,$$

define a basis in $\underline{M}_h$ with $N3 = NM1 + NM2 = \dim \underline{M}_h$

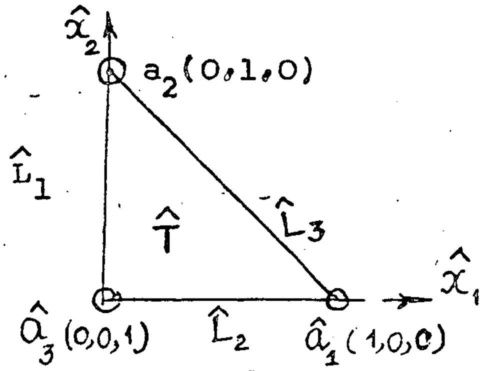
In (i)-(iii) we have already outlined the usual finite element procedure of construction of basis functions in $\underline{V}_T^*$, W_T, $\underline{M}_{\partial T}$ $\forall T \in T_h$, where $\forall T \in T_h$, $\underline{V}_T^*$ (resp. W_T) is the set of restrictions of functions in $\underline{V}_h^*$ (resp. W_h) to T, $\underline{M}_{\partial T}$ being the set of restrictions of functions in $\underline{M}_h$ to ∂T, while constructing basis functions in $\underline{V}_h^*$, W_h, $\underline{M}_h$. But for the purpose of computations, it is sufficient to construct basis functions in $\underline{V}_{\hat{T}}^*$, $W_{\hat{T}}$, $\underline{M}_{\partial\hat{T}}$ for the reference triangle $\hat{T}$:

$$\hat{T} = \{\hat{x}: \hat{x} = (\hat{x}_1, \hat{x}_2), 0 \leq \hat{x}_1, \hat{x}_2 \leq 1, 0 \leq \hat{x}_1 + \hat{x}_2 \leq 1\}$$

with boundary $\partial\hat{T}$ exactly in the same way as done for the generic triangle $T \in T_h$, keeping in mind that for $\hat{T}$, $\hat{\lambda}_1 = \hat{x}_1$, $\hat{\lambda}_2 = \hat{x}_2$,

$\hat\lambda_3 = 1-\hat x_1-\hat x_2$, $\{\hat\lambda_i\}_{i=1}^{3}$ being the barycentric coordinates for $\hat T$. Then, introducing invertible affine mappings

$F_T: \hat T \longrightarrow T$, $F_T^{-1}: T \longrightarrow \hat T$ $\forall T \in T_h$ and using their well known properties [12], one gets basis functions in $\underline{V}_T^*$, W_T, $\underline{M}_{\partial T}$ $\forall$ generic triangle $T \in T_h$.



For example, the basis functions $\{\hat{\underline{\Phi}}^i\}_{i=1}^{\hat N1}$, $\{\hat\chi^j\}_{j=1}^{\hat N2}$, $\{\hat{\underline{\mu}}^\ell\}_{\ell=1}^{\hat N3}$ in $\underline{V}^*_{\hat T}$, $W_{\hat T}$, $\underline{M}_{\partial\hat T}$, where $\hat N1 = 3(k+2)(k+1)/2$, $\hat N2 = 3+k(k-1)/2$, $\hat N3 = 3(k+1)+3k$, are given by:

$$\partial\hat T = \hat L_1 \cup \hat L_2 \cup \hat L_3\ ,\ \hat L_i \text{ is defined by } \hat\lambda_i = 0,\ 1 \le i \le 3,$$

<u>For k =1</u>, $\hat N1 = 9$, $\hat N2 = 3$, $\hat N3 = 6+3 = 9$.

I. $$\begin{aligned} &\hat{\underline{\Phi}}^1 = (1,0,0,0), && \hat{\underline{\Phi}}^2 = (\hat x_1,0,0,0), && \hat{\underline{\Phi}}^3 = (\hat x_2,0,0,0),\\ &\hat{\underline{\Phi}}^4 = (0,1,1,0), && \hat{\underline{\Phi}}^5 = (0,\hat x_1,\hat x_1,0), && \hat{\underline{\Phi}}^6 = (0,\hat x_2,\hat x_2,0)\\ &\hat{\underline{\Phi}}^7 = (0,0,0,1), && \hat{\underline{\Phi}}^8 = (0,0,0,\hat x_1), && \hat{\underline{\Phi}}^9 = (0,0,0,\hat x_2), \end{aligned} \tag{7.25}$$

II. $$\hat\chi^i = \hat\lambda_i\ ,\ 1 \le i \le 3. \tag{7.26}$$

III. $$\hat{\underline{\mu}}^1 = \begin{cases} (3\hat x_2-1,0) & \text{on } \hat L_1,\\ (0,0) & \text{on } \hat L_2 \cup \hat L_3 \end{cases};\quad \hat{\underline{\mu}}^2 = \begin{cases} (2-3\hat x_2,0) & \text{on } \hat L_1,\\ (0,0) & \text{on } \hat L_2 \cup \hat L_3 \end{cases};$$

$$\hat{\underline{\mu}}^3 = \begin{cases} (2-3x_1,0) & \text{on } \hat L_2,\\ (0,0) & \text{on } \hat L_1 \cup \hat L_3 \end{cases};\quad \hat{\underline{\mu}}^4 = \begin{cases} (3\hat x_1-1,0) & \text{on } \hat L_2,\\ (0,0) & \text{on } \hat L_1 \cup \hat L_3 \end{cases};$$

$$\hat{\underline{\mu}}^5 = \begin{cases} (3\hat{x}_1-1,0) & \text{on } \hat{L}_3, \\ (0,0) & \text{on } \hat{L}_2 \cup \hat{L}_1 \end{cases}; \qquad \hat{\underline{\mu}}^6 = \begin{cases} (2-3\hat{x}_1,0) & \text{on } \hat{L}_3, \\ (0,0) & \text{on } \hat{L}_2 \cup \hat{L}_1 \end{cases};$$

$$\hat{\underline{\mu}}^7 = \begin{cases} (0,1) & \text{on } \hat{L}_1 \\ (0,0) & \text{on } \hat{L}_2 \cup \hat{L}_3 \end{cases}; \qquad \hat{\underline{\mu}}^8 = \begin{cases} (0,1) & \text{on } \hat{L}_2 \\ (0,0) & \text{on } \hat{L}_3 \cup \hat{L}_1 \end{cases};$$

$$\hat{\underline{\mu}}^9 = \begin{cases} (0,1) & \text{on } \hat{L}_3, \\ (0,0) & \text{on } \hat{L}_1 \cup \hat{L}_2 \end{cases}. \tag{7.27}$$

Following the procedures developed in this section for the solution of the new scheme (Q_h^*) (instead of the scheme (Q_h)), numerical experiments have been carried out on the following problems, the final results of which will be presented here.

I. <u>BIHARMONIC (STOKES) PROBLEM</u> (Example (4.1)):

$\Lambda u \equiv \Delta\Delta u = f$ in Ω, $u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0$,

Data: $\Omega = (0,1)\times(0,1)$, $\overline{\Omega} = \Omega \cup \Gamma = [0,1]\times[0,1]$,

$f(x_1,x_2) = 24(x_1^2(x_1-1)^2 + x_2^2(x_2-1)^2) + 8(6x_1^2 - 6x_1 + 1)x$

$(6x_2^2 - 6x_2 + 1)\ \forall (x_1,x_2) \in \Omega$ [11].

The <u>exact solution</u> of the problem is: $u(x_1,x_2) = x_1^2 x_2^2 (x_1-1)^2 (x_2-1)$

(7.28)

The results of the numerical experiment are given in the Table 7.1.

II. <u>BENDING PROBLEMS OF CLAMPED ELASTIC PLATES</u>:

i) <u>ISOTROPIC CASE</u> (Example (4.2)): thickness h = const.,

$\Lambda u \equiv D\Delta\Delta u = f$ in Ω, $u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0$,

<u>Data</u>: $\Omega = (-1/2,1/2)\times(-3/4,3/4)$, $\overline{\Omega} = [-1/2,1/2]\times[-3/4,3/4]$,

$f = q$ = const., $\nu = 0.3$, $D = Eh^3/(12(1-\nu^2))$,

<u>The Timoshenko solution</u> [28] of the problem gives:

$u(0,0) = \alpha_T\, q/D$; $\psi_{ii}(0,0) = \beta_{iT}\, q\ (i = 1,2)$;

$$\psi_{11}(1/2,0) = \gamma_{1T}\, q; \quad \psi_{22}(0,3/4) = \gamma_{2T}\, q; \tag{7.29}$$

where the values of α_T, β_{iT}, γ_{iT} $(i = 1,2)$ are those given alongwith with the results of numerical experiment for the isotropic case in Table 7.2.

(ii) <u>ORTHOTROPIC CASE</u> (Example (4.3)): thickness h = const.,

$$\Lambda u \equiv D_1 u_{,1111} + 2H\, u_{,1122} + D_2 u_{,2222} = f \text{ in } \Omega,$$

$$u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0,$$

Data: $\Omega = (-1/2, 1/2)\mathrm{x}(-3/4, 3/4)$, $\bar{\Omega} = [-1/2, 1/2]\mathrm{x}[-3/4, 3/4]$, $f = q$ = const., $h = 0.01$, $E_1 = 0.21\mathrm{x}10^6$, $E_2 = 0.16\mathrm{x}10^6$, $G = 0.42\mathrm{x}10^5$ (all are to be taken in proper units of measurement), $\nu_2 = 0.07$, $E_1\nu_2 = E_2\nu_1$,

The <u>Szilard's solution</u> [27] of the problem gives:

$$u(0,0) = \alpha_s\, q, \; \psi_{ii}(0,0) = \beta_{is}\, q \; (i = 1,2), \; \psi_{11}(1/2,0) = \gamma_{1s} q,$$

$$\psi_{22}(0,3/4) = \gamma_{2s}\, q, \qquad (7.30)$$

where the values of α_s, β_{is}, γ_{is} $(i = 1,2)$ are those given alongwith the results of the numerical experiments for the orthotropic case in the Table 7.3.

<u>Remark (7.3)</u>: Although the results of Szilard (7.30) given in the Table (7.3) are themselves not very accurate, these have been included here just for the sake of some comparison of the results of the numerical experiment.

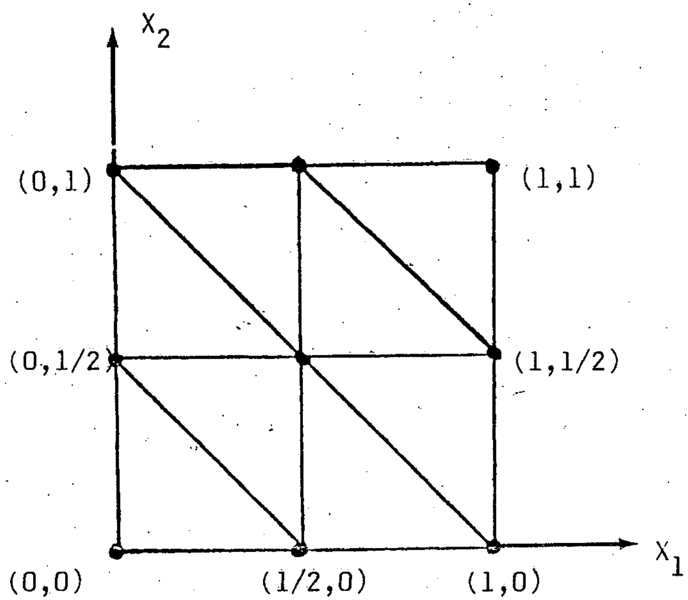
In all the numerical experiments and in the preparation of Tables (7.1)-(7.3), the following strategies have been adopted for the sake of simplicity in computations:

(i) $\bar{\Omega}$ has been triangulated into isosceles triangles as shown in the Figs. 7.1 and 7.2.

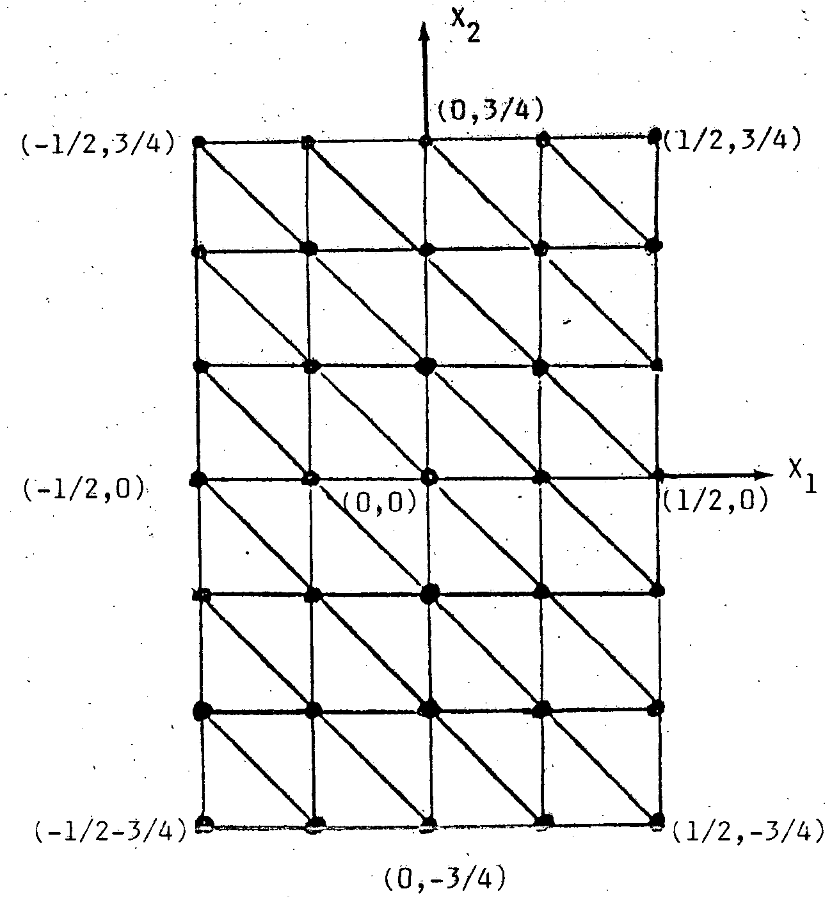
(ii) The basis functions given in (7.25)-(7.27) for $k = 1$ have been used alongwith usual finite element procedures of computation.

(iii) The extrapolated values of ψ_{hii} $(i = 1,2)$ at the nodes $a_h^j = (x_1^j, x_2^j)$ given in the Tables (7.1)-(7.3) have been determined simply by taking the arithmetic mean of the values $\psi_{hii}|_T\,(a_h^j)$ for all T containing the common node a_h^j.

ND1 = NO OF SUBDIVISIONS IN X_1- DIRECTION
ND2 = NO OF SUBDIVISIONS IN X_2- DIRECTION.



ND1 = ND2 = 2
NO OF TRIANGLES = 8
Fig. 7.1 : <u>BIHARMONIC CASE</u>



ND1 = 4, ND2 = 6
NO OF TRIANGLES = 48
Fig. 7.2 : <u>ISOTROPIC & ORTHOTROPIC CASES</u>

(iv) The global matrix $[K]$ in (7.20) has been constructed by assembling the element matrices $[K_T]$ $\forall T \in T_h$ as done in the standard finite element computations after applying (7.17) and (7.18) at the element level.

(v) The equation (7.19) has been solved by the Cholesky method of decomposition.

(vi) Only the points of $\overline{\Omega}$ at which 'u_h' and 'ψ_{hii}' $(i = 1,2)$ attain optimal values have been considered in the Tables (7.1)-(7.3).

Remark (7.4): For the sake of simplicity we have made the extrapolation of $\psi_{hii}(i = 1,2)$ at the nodal points $a_h^j = (x_1^j, x_2^j)$ by taking the arithmetic mean of $\psi_{hii}|_T (a_h^j)$ with $a_h^j \in T$ as stated in (iii). But one may apply better and more sophisticated methods to obtain improved results of extrapolation (see pages 283-284 of [29]).

TABLE (7.1): BIHARMONIC(STOKES) PROBLEM

$\psi_{hii} = \kappa_{hii} = u_{h,ii}$ $(i = 1,2)$, $\omega_h = -(\kappa_{h11} + \kappa_{h22})$

k = 1

ND1= ND2	No of* unknowns	$u_h(1/2,1/2)$	$\kappa_{h11}(1/2,1/2)$	$\kappa_{h11}(0,1/2)$	$\omega_h(1/2,1/2)$
2	25	0.005327	-0.07873	0.07473	0.1575
4	129	0.003662	-0.06652	0.10991	0.1330
EXACT SOLUTION (7.28)		0.003906	-0.0625	0.125	0.125

TABLE (7.2): CLAMPED ISOTROPIC PLATE PROBLEM

$u_h(0,0) = \alpha_h^I\ q/D$, $\psi_{hii}(0,0) = \beta_{ih}^I\ q(i = 1,2)$, $\psi_{h11}(1/2,0) = \gamma_{1h}^I\ q$,
$\psi_{h22}(0,3/4) = \gamma_{2h}^I q$,

k = 1

ND1	ND2	No of* unknowns	α_h^I	β_{1h}^I	β_{2h}^I	γ_{1h}^I	γ_{2h}^I
4	6	201	0.002256	-0.04181	-0.02230	0.06950	0.04894
TIMOSHENKO SOLUTION (7.29)			0.00220 (α_T)	-0.0368 (β_{1T})	-0.0203 (β_{2T})	0.0757 (γ_{1T})	0.0570 (γ_{2T})

TABLE (7.3): CLAMPED ORTHOTROPIC PLATE PROBLEM

$u_h(0,0) = \alpha_h^O\ q$, $\psi_{hii}(0,0) = \beta_{ih}^O\ q(i = 1,2)$, $\psi_{h11}(1/2,0) = \gamma_{1h}^O\ q$,

$\psi_{h22}(0,3/4) = \gamma_{2h}^O\ q$,

k = 1

ND1	ND2	No of* unknowns	α_h^O	β_{1h}^O	β_{2h}^O	γ_{1h}^O	γ_{2h}^O
4	6	201	0.1342	-0.04122	-0.01072	0.07197	0.04224
SZILARD'S RESULTS (7.30)			0.1529 (α_s)	-0.04443 (β_{1s})	-0.01761 (β_{2s})	0.08618 (γ_{1s})	0.02918 (γ_{2s})

*No of unknowns in Tables 7.1-7.3 do not include those eliminated at t[illegible] element level.

•All numerical values in Tables (7.1)-(7.3) are to be understood in proper units of measurement.

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